3rd International Conference

GEOMETRY AND TOPOLOGY OF MANIFOLDS

Krynica, 29 April – 5 May, 2001

Published by INSTITUTE OF MATHEMATICS of the Technical University of ŁÓDŹ

Contents

FOREWORD

This is the conference of the cycle initiated in 1998 with a meeting in Konopnica (`http://im0.p.lodz.pl/konferencje`) and is organized as in 1999 in Krynica from 29.04.2001 to 5.05.2001 (Poland). Krynica is a well known resort situated in Beskidy Mountains.

The main purpose of the conference is to present an overwiew of principal directions of research conducted in differential geometry, topology and analysis on manifolds and their applications, mainly (but not only) to Lie algebroids and related topics.

We would like to attract attention to:

- Riemannian, symplectic and Poisson manifolds,
- Lie groups, Lie groupoids, Lie algebroids and Lie-Rinehart algebras,
- foliations,
- characteristic classes.

The organizers of the conference are grateful to the following sponsors:

- Rector of the Technical University of Łódź,
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ABSTRACTS. PAPERS

THE STRUCTURE INVARIANTS OF A TRANSVERSE G-STRUCTURE

IVAN BELKO

Abstract

The problems of equivalence and integrability of G-structure were considered by many authors. Ch.Ehresmann, E.Cartan, S.Chern, V.Guillemin, D.Bernard and many others studied this problems. The solution of the equivalence problem is based on construction of structure invariants and cohomology classes correspondent. It is possible to pick out two main approaches to construction of the structure tensor, wich are different in their methods. We can call one of them geometrical, it uses the fundamental form and the forsion of the connection on principal bundle of the frames. Another approach is caracterised by use of differential operators and their Spencer cohomologies. The Ngo van Que paper [1] is an example of this.

Our goal is the construction of a structure invariants for a kth order foliated transversal structure on foliated manifold. Geometrical construction of such tensor for the first- order transversal structure is given in R.Wolak paper [2].

The particularity of our approach is in using of foliated Lie algebgoid anddevelopment of Spencer cohomology method to transversal differential operators. For a foliated manifold (B, F) the transversal k-jets of local diffeomorfisms forms the foliated Lie groupoid $\Pi^k(B)$. His Lie algebroid can be identified with the foliated Lie algebroid $\hat{J}^k(TB)$ of transversal k-jets of foliated vector filds. In this Lie algebroid the partial plate connection $\tau : TF \to \hat{J}^k(TB)$ is defined by canonical way. The natural projection by $Ker\ \tau$ determines the exact sequance of foliated vector bundles

$$0 \to \hat{J}^k_o(TB) \to \hat{J}^k(TrB) \to TrB \to 0.$$

The adjoint Lie algebra bundle $\hat{J}^k_o(TB) = \hat{J}^k_o(TrB)$ plays an important role in construction of a structure invariants.

A regular infinitesimal transversal kth order structure on a foliated manifold is a regular section of the bundle associated to Lie groupoid $\Pi^k(B)$. Such a bundle is foliated and permits to distinguish the foliated sections and structures.

A regular structure S defines a Lie subgroupoid $\Pi^k(S)$ in $\Pi^k(B)$. His Lie subalgebroid is a vector bundle and can be considered as a system of linear differential equations

$$E^k(S) \subset \hat{J}^k(TB).$$

On the base of the exacte sequance of vector bundles is built the transversal Spencer operator

$$\hat{D} : \Gamma(\hat{J}^k TB) \to \Gamma(\hat{J}^{k-1} TB \otimes Tr^* B).$$

This operator defines a transversal cohomologycal sequance

$$\delta : TrB \otimes S^{k+1}(Tr^*B) \otimes \Lambda^p(Tr^*B) \to TrB \otimes S^k(Tr^*B) \otimes \Lambda^{p+1}(Tr^*B).$$

In the terms of these cohomologies is defined the type and the degree of a structure S. For a connection in Lie algebroid $\hat{J}^k(TB)$ also in its Lie subalgebroid $E^k(S)$ can be defined the torsion. The class of transversal δ-cohmology correspondent is an obstacle to the integrability of a foliated first-order G-structure S.

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HOMOLOGY OF COMPLETE SYMBOLS AND NON-COMMUTATIVE GEOMETRY

MOULAY–TAHAR BENAMEUR and VICTOR NISTOR

Abstract

We identify the periodic cyclic homology of the algebra of complete symbols on a differential grupoid $\mathcal{G}$ in terms of the cohomology of $S^*(\mathcal{G})$, the cosphere bundle of $A(\mathcal{G})$, where $A(\mathcal{G})$ is the Lie algebroid of $\mathcal{G}$. We also relate the Hochschild homology of this algebra with the homogenous Poisson homology of the space $A^*(\mathcal{G}) \setminus 0 \cong S^*(\mathcal{G}) \times (0, \infty)$, the dual of $A(\mathcal{G})$ with the zero section removed. We use then these results to compute the Hochschild and cyclic homologies of the algebras of complete symbols associated with manifolds with corners, when the corresponding Lie algebroid is rationally isomorphic to the tangent bundle.

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METRIC-POLYNOMIAL STRUCTURES FROM GRAVITATIONAL LAGRANGIANS

ANDRZEJ BOROWIEC

Abstract

We study these metric-polynomial structures on manifold which arises as extremals of the Palatini variational principle for some class of gravitational Lagrangians. They can be described in the following way.

Let (M, g, Γ) be a n-dimensional (pseudo-) Riemannian manifold (M, g) equipped with a symmetric (i.e. torsion-free) connection Γ. Define a $(1,1)$ tensor field concomitant

$$S^\mu_\nu \equiv S^\mu_\nu(g, \Gamma) \doteq g^{\mu\alpha} R_{(\alpha\nu)}(\Gamma)$$

where $R_{(\alpha\nu)}(\Gamma)$ denotes the symmetric part of the Ricci tensor of Γ. Consider a family of scalar-valued concomitant

$$s_k \equiv s_k(g, \Gamma) \doteq \operatorname{tr} S^k$$

the so-called Ricci scalar of order k, $k = 1, \cdots, n$. For F being an arbitrary (differentiable) real-valued function of n-variables one can define the corresponding Lagrangian of the Ricci type

$$L_F \doteq \sqrt{g}\, F(s_1, \cdots, s_n)$$

Applying now the Palatini variational principle we arrive to the following results:

- the connection Γ is a Levi-Civita connection for some pseudo-Riemannian Einstein metric h,
- the $(1,1)$ tensor field S satisfies a polynomial equation

  $$w_F(S) = 0$$

  for some polynomial function $w_F(t)$ of constant coefficients,
- the metric h and polynomial structure S are compatible in a sense

  $$h(SX, Y) = h(X, SY)$$

  for each pair of tangent vector fields (X, Y) on M.

In particular, for f being a function of one variable the Lagrangian $L_f = \sqrt{g}\, f(s_1)$ reconstructs the Einstein theory. For $L_f = \sqrt{g}\, f(s_2)$, besides the Einstein equation, one gets a pseudo-Riemannian almost product structure ($S^2 = I$) and/or an almost-complex anti-Hermitian structure ($S^2 = -I$). Some other examples will be also considered.

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APPLICATION OF LIE ALGEBROIDS ON FIELD THEORIES

JAIME CAMACARO

Abstract

We will show some examples of how the Lie algebroids can play an interesting role in the study of gauge fields theories. Our talk will be based on four examples: Yang–Mills theory, Topological sigma models, Poisson sigma models and open bosonic string, where the Lie algebroid structure can be easily recognized and used to obtain an algebra which take in consideration the gauge structure.

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Thanks: JRC acknowledges the financial support of the Agencia Española de Cooperación Internacional, under an AECI scholarship

Keywords: BV master equation, Gauge theories, Lie algebroids

ON A CLASS OF RICCI-RECURRENT MANIFOLDS

STANISŁAW EWERT-KRZEMIENIEWSKI

1 Introduction

Following Prvanivić ([P]), a semi-Riemannian manifold $(M,\ g)$ will be called conformally quasi-recurrent if its Weyl conformal curvature tensor C satisfies

$$\begin{aligned}\nabla_Z C(X,Y,V,W) = w(Z)C(X,Y,V,W)+ \\ p(X)C(Z,Y,V,W) + p(Y)C(X,Z,V,W)+ \\ p(V)C(X,Y,Z,W) + p(W)C(X,Y,V,Z)\end{aligned} \tag{1.1.1}$$

for some 1-forms w, p. In condition considered originally by Prvanović $w = 2p$. However, the last relation together with (1.1.1) implies that for the tensor C (and in fact for any generalized curvature tensor satisfying (1.1.1)) the second Bianchi identity must hold. The aim of this note is to give a classification of conformally quasi-recurrent manifolds in the sense of (1.1.1) which are simultaneously Ricci-recurrent, i. e. those the Ricci tensor S satisfies

$$\nabla S = b \otimes S$$

for some 1-form b.

For a generalized curvature tensor B define $\widetilde{B}$ by

$$g\left(\widetilde{B}\,(X,Y)\,V,W\right) = B\,(X,Y,V,W)\,.$$

Then for a $(0,k)$ tensor field T, $k \geq 1$, and $(0,2)$ tensor field S we define the tensor fields $B \cdot T$ and $Q(S,T)$ by the formulas

$$\begin{aligned}&(B\cdot T)\,(X_1,...X_k;X,Y) = \\ &\quad -T\left(\widetilde{B}\,(X,Y)\,X_1,X_2,...,X_k\right) - ... - T\left(X_1,...,X_{k-1},\widetilde{B}\,(X,Y)\,X_k\right),\end{aligned}$$

$$\begin{aligned}&Q\,(S,T)\,(X_1,...X_k;X,Y) = \\ &\quad T\,((X\wedge_S Y)\,X_1,X_2,...,X_k) + ... + T\,(X_1,...,X_{k-1},(X\wedge_S Y)\,X_k)\,,\end{aligned}$$

where

$$(X\wedge_S Y)\,Z = S(Y,Z)X - S(X,Z)Y.$$

If the tensors $R \cdot R$ and $Q(S,R)$ are linearly dependent then the manifold is said to be Ricci-generalized pseudosymmetric one ([D-D]). It is obvious that any semisymmetric as well as any Ricci flat manifold is Ricci generalized pseudosymmetric. The manifold (M,g) is Ricci-generalized pseudosymmetric iff the relation

$$R\cdot R = LQ(S,R) \tag{1.1.2}$$

holds on the set $\{x \in M, Q(S,R)\,(x) \neq 0\}$, L being a function on M. Remark that the relation (1.1.2) with $L = 1$ is of particular importance.

All manifolds under consideration are assumed to be smooth Hausdorff connected and their metrics are not assumed to be definite.

2 Results

The first Lemma shows the difference between the 1-forms w and p :

Lemma 2.1. *Suppose that at a point of the manifold M relation (1.1.1) holds. Then*

$$p_r C^r_{ijk} = 0,$$

$$w_r C^r_{ijk} = C^r{}_{ijk,r},$$

$$C_{hijk,[lm]} = \Delta w_{lm} C_{hijk} + p_{hm} C_{lijk} + p_{im} C_{hljk} + p_{jm} C_{hilk} + p_{km} C_{hijl} - \\ p_{hl} C_{mijk} - p_{il} C_{hmjk} - p_{jl} C_{himk} - p_{kl} C_{hijm},$$

where $\Delta w_{lm} = w_{l,m} - w_{m,l}$, $p_{hm} = p_{h,m} - p_h p_m$ and comma denotes covariant differentiation.

From the so called Patterson identity we have

Proposition 2.2. *Let M be a 4-dimensional manifold with nowhere vanishing Weyl conformal curvature tensor C. If C satisfies (1.1.1), then M is conformally recurrent manifold, precisely*

$$\nabla C = (w + 2p) \otimes C.$$

In the sequel we shall assume the following hypothesis:

(H) M *is a Ricci – recurrent manifold with nowhere vanishing Weyl conformal curvature tensor and Ricci tensor. Moreover, the Weyl conformal curvature tensor satisfies* (1.1.1), *p does not vanishes on a dense subset and the Ricci tensor is not parallel.*

By hypothesis, M admits a covector field b satisfying

$$S_{ij,k} = b_k S_{ij}, \quad S_{ij,kl} = (b_{k,l} + b_k b_l) S_{ij}, \quad S_{ij,[kl]} = \Delta b_{kl} S_{ij},$$

where $\Delta b_{kl} = b_{k,l} - b_{l,k}$.

Proposition 2.3. *Let M ($dim M > 3$) be a Ricci-recurrent manifold with non-parallel Ricci tensor and suppose that M admits a covector field p with properties:*
i) p does not vanish on a dense subset of M;
ii) $p_r C^r_{ijk} = 0$ on M.
Then the scalar curvature of M vanishes.

Lemma 2.4. *Under hypothesis (H) relations*

$$p_r S^r_h = 0,$$

$$db = 0$$

$$S_{mr} C^r_{ijk} = 0$$

hold on M.

From the above Lemma it follows

Corollary 2.5. *Under hypothesis* (H)

$$d(w+2p)=0$$

holds on M.

Making use of the Corollary 2.5 one can deduce

Proposition 2.6. *Assume that on a manifold* $(M,\,g)$ *hypothesis* (H) *is satisfied.*

If $t=w-2p=0$ *on* $(M,\,g)$, *then the manifold is conformally related to a non-conformally flat conformal symmetric one* $(M,\,Exp\,(2f)\,g)$. *Conversely, if* $(M,\,g)$ *is conformally related to a non-conformally flat conformally symmetric one, then* $w-2p=0$.

Proposition 2.7. *Suppose that on a manifold* M, $(dimM>4)$, *hypothesis* (H) *is satisfied. If* $b_l\neq 0$ *at a point* $x\in M$, *then on some neighbourhood of* x *there exists null (i.e. isotropic) parallel vector field*

$$v_i=Exp[\frac{-1}{2}b]\;k_i,\qquad \partial_i b=b_i,$$

related to the Ricci tensor by

$$S_{ij}=\epsilon\;k_ik_j,\qquad |\epsilon|=1.$$

Define the tensor

$$\begin{aligned}Q(S,C)_{hiqtpj}=&\,S_{hp}C_{jiqt}-S_{hj}C_{piqt}+S_{ip}C_{hjqt}-S_{ij}C_{hpqt}+\\&\,S_{qp}C_{hijt}-S_{qj}C_{hipt}+S_{tp}C_{hiqj}-S_{tj}C_{hiqp}.\end{aligned}$$

It is well known, that if the scalar curvature vanishes and the rank of the Ricci tensor is one, then

$$Q(S,C)_{hiqtpj}=Q(S,R)_{hiqtpj}$$

Lemma 2.8. *Suppose that on a manifold* M, $(dimM>4)$, *hypothesis* (H) *is satisfied. Then*

$$t_rb^r\cdot Q(S,C)_{hiqtpj}=0.$$

Lemma 2.9. *Suppose that on a manifold* M, $(dimM>4)$, *hypothesis* (H) *is satisfied and*

$$t_rb^r=0.$$

Then

$$(n-4)\,b_rb^r=0,$$

$$t_lC_{hijk}+t_jC_{hikl}+t_kC_{hilj}=0$$

and

$$t_l\left(C_{tqlr}C^r_{jih}-C_{tqjr}C^r_{lih}\right)=0$$

Proposition 2.10. *Suppose that on a manifold M, $(dimM \geq 4)$, hypothesis (H) is satisfied. If at $x \in M$, $t_r b^r(x) \neq 0$ (hence $b_l(x) \neq 0$), then on some neighbourhood of x the Riemann-Christoffel curvature tensor has the form*

$$R_{hijk} = k_i k_j S_{hk} - k_i k_k S_{hj} + k_h k_k S_{ij} - k_h k_j S_{ik},$$

where $S_{ij} = z^r z^s R_{rijs}$, $z^r k_r = 1$.

As a consequence of Propositions 2.3, 2.7 and 2.10 we obtain

Corollary 2.11. *Suppose that on a manifold M, $(dimM > 4)$, hypothesis (H) and and $Q(S,C)_{hiqtpj} = 0$ hold. If $b_l \neq 0$ at a point $x \in M$, then on some neighbourhood of x the metric g is of the Walker type. Moreover, M is semi-symmetric, i.e. $R \cdot R = 0$.*

On the other hand, if $Q(S,C)_{hiqtpj} \neq 0$, then Lemma 2.9 results in

Corollary 2.12. *Let on a manifold M, $(dimM > 4)$, hypothesis (H) be satisfied. Suppose moreover that $Q(S,C)_{hiqtpj} \neq 0$ and $w - 2p \neq 0$. Then*

$$(R \cdot R)_{hiqtpj} = Q(S,R)_{hiqtpj}.$$

Thus we conclude with the following

Theorem 2.13. *A conformally quasi-recurrent and Ricci-recurrent manifold of dimension $n > 4$ with nowhere vanishing Weyl conformal curvature tensor and Ricci tensor non-conformally related to a conformally symmetric one must be necessary Ricci-generalized pseudosymmetric manifold.*

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ON INVARIANTS OF CONTINUOUS SUBGROUPS OF THE GENERALIZED POINCARÉ GROUP $P(1,4)$

VASYL FEDORCHUK

Abstract

For all continuous subgroups of the group $P(1,4)$ the invariants in the five-dimensional Minkowski space $M(1,4)$ have been constructed. The invariants obtained are one-, two-, three and four-dimensional.

On the base of the invariants obtained the nonsingular manifolds in the spaces $M(1,3)\times \mathbf{R}$ and $M(1,4)\times \mathbf{R}$ invariant under nonconjugate subgroups of the group $P(1,4)$ have been described.

The manifolds obtained have already been used for the symmetry reduction of some important equations of the theoretical physics in the spaces $M(1,3)\times \mathbf{R}$ and $M(1,4)\times \mathbf{R}$.

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INTEGRATING LIE ALGEBROIDS TO LIE GRUPOIDS

RUI LOJA FERNANDES

Abstract

I will report on some recent joint work with Marius Crainic (Utrecht), where we give the obstructions for integrating Lie algebroids to Lie grupoids [2]. This puts into a new perspective the work of Cattaneo and Felder [1] for the special case of Poisson manifolds and the "new" proof of Lie's third theorem given by Duistermaat and Kolk in [3].

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JACOBI STRUCTURES REVISITED

JANUSZ GRABOWSKI

Abstract

A lifting procedure of first-order multi-differential operators is defined which maps the Nijenhuis-Richardson into the Schouten bracket. This is a way of associating canonically a Lie algebroid with any local Lie algebra structure on a 1–dimensional vector bundle.

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DIFFERENTIAL GEOMETRY STRUCTURE FOR FORMAL MAPS

ROMAN KADOBIANSKI and VITALY KUSHNIREVITCH

Abstract

Usually, differential geometry is formulated in the terms of smooth maps of smooth manifolds. The conception proposed by I.M.Gel'fand and Yu.L.Daletskii is to replace smooth manifolds by formal one. The formal manifold is the pair $(\mathfrak{A}, M)$, where $\mathfrak{A}$ is Lie algebra and M is a module over it. Let $L_k(X,Y)$ be k-linear map from X to Y; $(X,Y) = \prod_{k=1}^{\infty} L_k(X,Y)$. Formal map is defined as sequence $a = (a_1, a_2, \dots) \in (X,Y)$, $a_k \in L_k(X,Y)$. The natural further step is to use formal maps instead of smooth ones. It turns out that Lie algebra structure and module over it can be defined in the space of formal maps. So, differential geometry of formal maps can be constructed. For example, such construction is useful in nonlinear partial diferential equations theory and for different nontrivial dynamical systems in physics.

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GENERALIZED SYMMETRIC SPACES

OLDŘICH KOWALSKI

Abstract

1 Introduction

Let (M, g) be a Riemannian symmetric space. Then for any $x \in M$, there exists an isometry $s_x : M \longrightarrow M$ such that x is an isolated fixed point of s_x and ${s_x}^2 = \mathrm{Id}$. Then we have

$$(s_x)_{*x} = (-\mathrm{Id})_x$$
$$v_x \mapsto -v_x$$

Now we consider a generalization of the notion of Riemannian symmetric spaces: Let (M, g) be a Riemannian manifold. We consider only the following condition.

$$\forall x \in M, \exists s_x : M \longrightarrow M : \text{isometry} \quad \text{s.t. } x : \text{isolated fixed point}$$

Such an isometry s_x is called a **generalized symmetry**. And a set of generalized symmetries $\{s_x \mid x \in M\}$ is called a (Riemannian) ***s*-structure**.

Lemma 1.1. *If $\{s_x \mid x \in M\}$ on (M, g) is an s-structure, then the closure $Cl(\{s_x\})$ of the group generated by all s_x, $x \in M$, in the Lie group $I(M, g)$ (the full isometry group) acts transitively on (M, g).*

Theorem 1.1 (Brickel). *Each (M, g) admitting an s-structure is homogeneous and thus real analytic.*

Here, "homogeneous" means that $\forall x, y \in M$, $\exists \varphi : M \longrightarrow M$: isometry, such that $\varphi(x) = y$.

The proof is elementary and interesting, but not short.

Affine case analogue:

Theorem 1.2 (Ledger). *Let (M, ∇) be an affine manifold (with an affine connection ∇). Let there exists a family $\{s_x \mid x \in M\}$ of generalized affine symmetries. If the map $x \mapsto s_x$ is smooth, then (M, ∇) is a homogeneous affine manifold.*

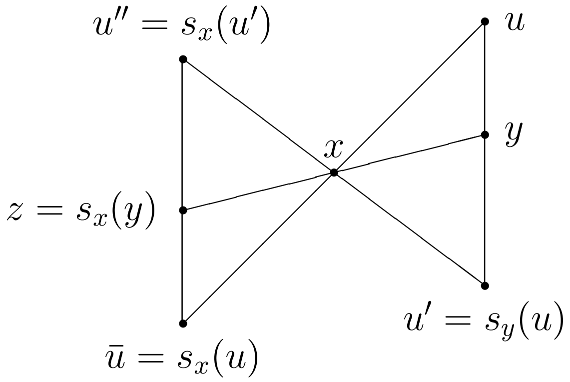
In general, it is an open problem whether (M, ∇) is still homogeneous if the map $x \mapsto s_x$ is not smooth. (Elementary but interesting question).

Regularity condition:

$$s_x \circ s_y = s_z \circ s_x \quad z = s_x(y) \quad \forall x, y \in (M, g)$$

This condition is satisfied for each symmetric space and its standard symmetries s_x!

Example. In the case $\mathbb{R}^2$, we can illustrate the above condition by a picture. (In general, it is not easy to check).



Definition 1. *Tangent symmetry field S of $\{s_x \mid x \in M\}$*

$$S_x := (s_x)_{*x} \quad \forall x \in (M, g)$$

Lemma 1.2. *The regularity condition for $\{s_x \mid x \in M\}$ is satisfied if and only if S is invariant with respect to each s_x:*

$$(s_x)_{*y} \circ S_y = S_{s_x(y)} \circ (s_x)_{*y} \quad \forall x, y \in (M, g)$$

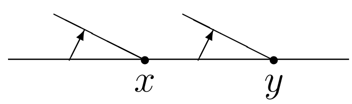
In the case of symmetric spaces,

$$S_y = (-\mathrm{Id})_y, \quad S_{s_x(y)} = (-\mathrm{Id})_{s_x(y)},$$

and the regularity follows.

An s-structure $\{s_x \mid x \in M\}$ which satisfies the regularity condition is called a **regular s-structure**. (It is a very strong condition).

Example of regular s-structures:
$M = \mathbb{R}^2$, $s_x =$ rotation arround x with constant angle α.
It is obvious from Lemma 2 that this s-structure is regular because each tangent symmetry S_x is also a rotation. (This direct proof of regularity is a good excercise for high school students).



Proposition 1.1. *If $\{s_x \mid x \in M\}$ is a regular s-structure on (M, g), then the tangent field is analytic and the map $(x, y) \mapsto s_x(y)$ is also analytic.*

Proposition 1.2. *If (M, g) admits an s-structure (resp. regular s-structure) $\{s_x \mid x \in M\}$, then (M, g) admits an s-structure (resp. regular s-structure) $\{s'_x \mid x \in M\}$ of finite order. It means that $\exists k \in \mathbb{Z},\ k \geq 2$ s.t.*

$$(s'_x)^k = Id \quad \textit{for } \forall x \in M.$$

Remark: It it not true for the affine case.

If there exists a regular s-structure $\{s_x \mid x \in M\}$ of order k on (M, g), then (M, g) is called a **k-symmetric space**. (Without regularity, a pointwise k-symmetric space).

Thus if there exists a regular s-structure on (M, g), then for some $\exists k$, (M, g) is a k-symmetric space.

Regularity condition is really an **additional** condition but in a nontrivial sense:
1) (R.A. Marinosci [4]) If $\dim(M, g) \leq 5$, then every (M, g) admitting an s-structure also admit a regular s-structure.
2) $S^9 = SU(5)/SU(4)$
S^9 is a geodesic sphere in (P^5, Fubini-Study metric) with the induced metric, which is not the standard sphere. This S^9 admits an s-structure of order 4 but no regular s-structure at all.

There are finitely many other examples $S^{13}, S^{17}, \cdots$.

In the following, we always assume regularity!

(M, g) is called a **generalized symmetric space** if it admits a regular s-structure.

(M, g) is called of **order** k $(k \geq 2)$ if it admits a regular s-structure of order k but not of any lower order $l < k$.

2 Examples and Classifications

Dimension $\boxed{n = 3}$

$$
\begin{aligned}
G &= \left\| \begin{matrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{matrix} \right\| \\
& \quad \text{(solvable)} \\
G &\cong \mathbb{R}^3(x, y, z)
\end{aligned}
$$

Special invariant metrics on G:

$$g = e^{2z}dx^2 + e^{-2z}dy^2 + \lambda^2 dz^2$$

where $\lambda > 0$ is an arbitrary parameter.
Symmetry of order 4 at the origin:

$$x' = -y, \quad y' = x, \quad z' = -z$$

(See B-Spaces by Takahashi).

Thses are all generalized symmetric spaces of dimension $n = 3$ which are not locally symmetric!

One kind of symmetric spaces:

G: compact connected Lie group
Consider the coset space $(G \times G)/\Delta(G \times G)$, where $\Delta(G \times G)$ is the **diagonal** of $G \times G$, i.e., $\Delta(G \times G) = \{(g, g) \mid g \in G\}$. $G \times G/\Delta(G \times G)$ is diffeomorphic to G via the map

$$(g_1, g_2) \mapsto g_1 {g_2}^{-1}.$$

$G \times G$ acts on G by

$$(g_1, g_2)(y) = g_1 y {g_2}^{-1}.$$

The isotropy group at the origin $e \in G$ is $\Delta(G \times G)$.

Now define $\sigma : G \times G \longrightarrow G \times G$ by

$$\sigma(g_1, g_2) = (g_2, g_2),$$

which is an involutive automorphism. The fixed point set is $(G \times G)^\sigma = \Delta(G \times G)$. σ induces a map $s : G \longrightarrow G$ defined by

$$s(g) = g^{-1} \quad \text{for } \forall g \in G.$$

Take a bi-invariant Riemannian metric Φ on G. Then Φ is **invariant** with respect to the action of $G \times G$ on G. It is also invariant with respect to $s : G \longrightarrow G$.

Then (G, Φ) is a Riemannian symmetric space in which the symmetry with respect to e is just the map

$$s : g \mapsto g^{-1}.$$

The symmetry s_x at general $x \in G$ is given by

$$g \mapsto xg^{-1}x.$$

Generalization(Ledger + Obata)

$$G^{k+1}/\Delta G^{k+1} \cong G^k$$

via $\pi : G^{k+1} \longrightarrow G^k$, where

$$\pi(g_1, \cdots, g_{k+1}) = (g_1 {g_{k+1}}^{-1}, \cdots, g_k {g_{k+1}}^{-1}).$$

Further define $\sigma : G^{k+1} \longrightarrow G^{k+1}$ by

$$\sigma(g_1, \cdots, g_{k+1}) = (g_{k+1}, g_1, \cdots, g_k).$$

Then σ is an automorphism of order $k + 1$. It induces a map $s : G^k \longrightarrow G^k$ defined by

$$s(g_1, \cdots, g_k) = ({g_k}^{-1}, g_1 {g_k}^{-1}, \cdots, g_{k-1} {g_k}^{-1}).$$

Let Φ be a bi-invariant metric on G. Then Φ generates a bi-invariant metric Φ^{k+1} on G^{k+1} such that

$$(G^{k+1}, \Phi^{k+1}) \cong \underbrace{(G, \Phi) \times \cdots \times (G, \Phi)}_{k+1}.$$

Then Φ^{k+1} induces a G^{k+1}-invariant metric $\Phi^{[k]}$ on G^k. The Riemannian manifold $(G^k, \Phi^{[k]})$ is a $(k + 1)$-symmetric space. (Here $\Phi^{[k]} \neq \Phi^k$). $(G^k, \Phi^{[k]})$ is not a symmetric space.

Proposition 2.1. *Let*
a) G be compact and simple,
b) $\Phi = -$(Killing form).
Assume that $\tau : G^{k+1} \longrightarrow I(G^k, \Phi^{[k]})$ has as image the full identity component of $I(G^k, \Phi^{[k]})$. Then $(G^k, \Phi^{[k]})$ is not l-symmetric for any $l < k + 1$.

Example. If we choose $G = SO(3)$, $\Phi = -$(Killing form), then the assumption of the Proposition 3 can be shown to be satisfied. Hence we get

Theorem 2.1. *For each $k \geq 2$, there is a generalized symmetric space (M, g) of order k, i.e., k-symmetric but not l-symmetric for $\forall l < k$. $M = G/K$ with semisimple group G.*

Classifications:
1) All generalized symmetric spaces of order 3 with semisimple (or reductive group) G by A. Gray [1].
2) All compact generalized symmetric spaces of order 4 by J.A. Jiménez [2].

An example of generalized symmetric Riemannian spaces of **solvable type**:

$$G = \left\| \begin{array}{ccccc} e^{u_0} & 0 & \cdots & 0 & x_0 \\ 0 & e^{u_1} & \cdots & 0 & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{u_n} & x_n \\ 0 & 0 & \cdots & 0 & 1 \end{array} \right\| \cong \mathbb{R}^{2n+1}$$

$u_0, u_1, \cdots, u_n, x_1, \cdots, x_n$ are variables, where

$$u_0 + u_1 + \cdots u_n = 0.$$

We define a Riemannian metric

$$g = \sum_{i=0}^{n} e^{-2u_i}(dx_i)^2 + a \sum_{\alpha,\beta=1}^{n} du_\alpha du_\beta, \quad a > 0.$$

Then
1) (G, g) is generalized symmetric of order $2n + 2$ (even).
2) The group of all isometries preserving the origin is finite and isometric to

$$(\mathbb{Z}_2)^{n+1} \times S_{n+1}$$

where S_{n+1} is the permutation group pf $n + 1$ elements. Thus, the group is of order $2^{n+1}(n + 1)!$.

Remark. There are also examples of generalized symmetric spaces of solvable type and odd order. For any order k, there exists an example of solvable type. This is the main obstacle to the classification of **all** generalized symmetric spaces (in contrary to the ordinary symmetric spaces, where the classification is known).

3 Canonical connection

Let (M, g) be a Riemannian manifold and $\{s_x \mid x \in M\}$ a regular s-structure on M. Then there is a unique affine conncetion $\widetilde{\nabla}$ on M such that
(i) $\widetilde{\nabla}$ is invariant under all s_x,
(ii) $\widetilde{\nabla} S = 0$ (where S is the tangent symmetry field defined by $S_x := (s_x)_{*x} \quad x \in M$).

In the explicit form, $\widetilde{\nabla}$ is given by the Ledger's formula

$$\widetilde{\nabla}_X Y = \nabla_X Y - (\nabla_{(I-S)^{-1}X} S)(S^{-1}Y) \qquad \text{for } X, Y \in \mathfrak{X}(M)$$

where ∇ denotes the Riemannian connection of (M, g).

Note that since S has no engenvalue equal to 1 (s_x has only isolated fixed point), $(I - S)^{-1}$ exists.

$\widetilde{\nabla}$ is always complete.

Each tensor field P on M which is invariant by all s_x is parallel with respect to $\widetilde{\nabla}$, i.e., $\widetilde{\nabla} P = 0$. Therefore, the torsion tensor field $\widetilde{T}$ and the curvature tensor field $\widetilde{R}$ of $\widetilde{\nabla}$ are parallel:

$$\widetilde{\nabla}\widetilde{T} = 0, \quad \widetilde{\nabla}\widetilde{R} = 0.(*)$$

Remark that if M is a symmetric space with standard $\{s_x\}$, $\widetilde{\nabla} = \nabla$.

Because of $(*)$, we have

$$\widetilde{\nabla}\widetilde{R} = \widetilde{\nabla}\widetilde{T} = \widetilde{\nabla} S = \widetilde{\nabla} g = 0.$$

Therefore we can linearize Riemannian manifolds $((M, g), \{s_x \mid x \in M\})$ with specific regular s-structure. We get algebraic objects $(V, <, >, S, \widetilde{R}, \widetilde{T})$, where S is a nonsingular linear isometry without fixed directions and $\widetilde{R}, \widetilde{T}$ are tensors of types $(1, 3)$, $(1, 2)$ respectively satisfying standard algebraic identities for the curvature and torsion. These objects are called **infinitesimal models**.

We have one-to-one correspondence between the set of simply connected Riemannian manifolds with regular s-structures and the set of infinitesimal models. Using the methods of linear algebra, we can classify regular s-structures and hence also generalized symmetric spaces (at least locally and for small dimensions).

If (M, g) is a generalized symmetric space and simply connected, then we have the de Rham decomposition

$$M = M_1 \times \cdots \times M_r,$$

here if $\{s_x\}$ is an s-tructure of order k of M, each M_i is a generalized symmetric space of order k_i and k_i divides k $(i = 1, \cdots, r)$.

4 Theory of eigenvalues for generalized symmetric spaces

This theory is useful for the classification procedure of generalized symmetric spaces.

Let $\{s_x \mid x \in M\}$ be a regular s-structure. Then $S_x : T_xM \longrightarrow T_xM$ has the same eigenvalues for $\forall x \in M$. S_x is real, orthogonal, without fixed vectors. Hence its system $(\theta_1, \cdots, \theta_n)$ of eigenvalues must satisfy

$$|\theta_i| = 1, \quad \theta_i \neq 1 \quad \text{for } i = 1, \cdots, n$$

and

$$(\bar{\theta_1}, \cdots, \bar{\theta_n}) = (\theta_1, \cdots, \theta_n) \qquad \text{up to numeration.}$$

Every n-tuple with these properties is called an **admissible** n-tuple. Denote by $\mathcal{P}_n$ the set of all admissible n-tuples.

We now make $\mathcal{P}_n$ a **partially ordered** set. We introduce **characteristic equations** (of two classes):

a) $X_iX_j = 1, \quad i, j = 1, \cdots, n,$

b) $X_iX_j = X_k, \quad i \neq j \neq k \neq i, \quad i, j, k = 1, \cdots, n.$

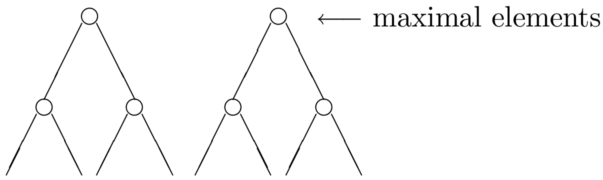
Each element $(\theta_1, \cdots, \theta_n) \in \mathcal{P}_n$ satisfies at least one equation of type a) $(n \geq 2)$. (If the eigenvalues of S_x on (M, g) do not satisfy any equation of type b), then (M, g) is locally symmetric.)

We introduce the **partial ordering** "$\preceq$" on $\mathcal{P}_n$ as follows: $(\theta_i) \preceq (\theta'_i)$ if all characteristic equations satisfied by (θ_i) are also satisfied by (θ'_i) after possible re-numeration.

(Here $(\theta_i) \preceq (\theta'_i)$ and $(\theta'_i) \preceq (\theta_i)$ is possible, e.g., if (θ'_i) is a renumeration of (θ_i).)

Proposition 4.1. *Let $\{s_x \mid x \in M\}$ be a regular s-structure on (M, g) with the corresponding eigenvalues (θ_i). Let $(\theta'_i) \in \mathcal{P}_n$ and $(\theta'_i) \succeq (\theta_i)$. Then there is a regular s-structure $\{s'_x \mid x \in M\}$ on (M, g) with the eigenvalues (θ'_i).*

Theorem 4.1. *In the partially ordered set $(\mathcal{P}_n, \preceq)$, there is only a finite set of maximal elements, which are all of finite order.*



This set will be denoted by $\mathcal{D}_n$.

The same procedure works without regularity condition.

We only have to add new type of characteristic equations

c) $X_i X_j X_k = X_l$.

The corresponding set $\mathcal{D}'_n$ of maximal elements contains $\mathcal{D}_n$, but it is bigger.

Now we go back to the regular s-structures. We have the following theorem.

Theorem 4.2. *For every dimension $n \geq 2$, there is a finite set $\mathcal{D}_n = \{(\theta_1^\alpha, \cdots, \theta_n^\alpha) \mid \alpha = 1, \cdots, r(n)\}$ with the following properties:*
a) all elements of $\mathcal{D}_n$ are of finite order,
b) if (M, g) admits a regular s-structure, then it also admits a regular s-structure with the system of eigenvalues contained in $\mathcal{D}_n$.

(Warning: the same (M, g) can admit regular s-structures corresponding to more elements of $\mathcal{D}_n$).

This theorem is important for the classification of generalized symmetric spaces in low dimensions.

4.1 Basic systems of eigenvalues in small dimensions

$\mathcal{D}_2$:	$(-1,-1)$	(symmetric space)
$\mathcal{D}_3$:	$(-1,-1,-1)$	(symmetric space)
	$(i,-i,-1)$	(4-symmetric space of dimension 3)
$\mathcal{D}_4$:	$(-1,-1,-1,-1)$	(symmetric space)
	$(\theta,\theta,\theta^2,\theta^2),\quad \theta = e^{\frac{2\pi i}{3}}$	($\exists$ new example of order 3)
	$(i,-i,-1,-1)$	(direct product or space of order 3)
	$(\theta,\theta^2,\theta^3,\theta^4),\quad \theta = e^{\frac{2\pi i}{5}}$	(only space of order 3)
$\mathcal{D}_5$:	$(-1,-1,-1,-1,-1)$	(symmetric space)
	$(i,-i,-1,-1,-1)$	(direct product)
	$(i,-i,i,-i,-1)$	($\exists$ new examples of order 4)
	$(\theta,\theta^2,\theta^3,\theta^4,\theta^5),\quad \theta = e^{\frac{2\pi i}{6}}$	($\exists$ new examples of order 6)
	$(e^{\frac{\pi i}{4}}, e^{-\frac{\pi i}{4}}, i, -i, -1)$	(no additional examples)

An estimate: Let $k(n)$ denote the maximum of all orders of the elements in $\mathcal{D}_n$. Then

$$\begin{aligned} k(n) &\leq 5^{\frac{n}{4}} && n: \text{even},\\ k(n) &\leq 2\cdot 5^{\frac{n-1}{4}} && n: \text{odd}. \end{aligned}$$

We see that $k(4)\leq 5$ (exact estimate) and $k(5)\leq 10$ (not exact, in fact $k(5)\leq 8$). It is a natural question whether there is an estimate to improve the above inequality or not.

4.2 Classification in dimension $n = 4$

All generalized symmetric spaces of dimension 4 are symmetric spaces or spaces of order 3 isometric to one of the following forms:

$$M = \left\| \begin{matrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{matrix} \right\| \Big/ \left\| \begin{matrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{matrix} \right\|$$

$$\text{where} \quad \det \left\| \begin{matrix} a & b \\ c & d \end{matrix} \right\| = 1$$

$$M = \left(\begin{matrix}\text{group of all equiaffine trans-}\\ \text{formations of the Euclidean}\\ \text{plane}\end{matrix} \right) \Big/ \left(\begin{matrix}\text{subgroup of all rotations ar-}\\ \text{round the origin}\end{matrix} \right)$$

$M \cong \mathbb{R}^4[x,y,u,v]$ and then

$$\begin{aligned} g = {} & (-x+\sqrt{x^2+y^2+1})du^2 + (x+\sqrt{x^2+y^2+1})dv^2 \\ & -2ydudv + \lambda^2 \left[\frac{(1+y^2)dx^2 + (1+x^2)dy^2 - 2xydxdy}{1+x^2+y^2} \right] \end{aligned}$$

where $\lambda > 0$ is a parameter.

A typical symmetry of order 3 at the origin is given by

$$\begin{aligned} u' &= \cos\frac{2\pi}{3}\cdot u - \sin\frac{2\pi}{3}\cdot v,\\ v' &= \sin\frac{4\pi}{3}\cdot u + \cos\frac{4\pi}{3}\cdot v,\\ x' &= \cos\frac{2\pi}{3}\cdot x - \sin\frac{2\pi}{3}\cdot y,\\ y' &= \sin\frac{4\pi}{3}\cdot x + \cos\frac{4\pi}{3}\cdot y. \end{aligned}$$

4.3 Classification in dimension $n = 5$

A generalized symmetric space of dimension 5 is locally isometric to one of the followings:

- a symmetric space,
- 11 families of order 4,
- 1 family of order 6.

4.3.1 Examples of order 4 – selection:

Example 1. Nilpotent matrix group

$$\left\| \begin{array}{cccc} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ u & v & 1 & z \\ 0 & 0 & 0 & 1 \end{array} \right\| \cong \mathbb{R}^5[x, y, z, u, v],$$

$$g = dx^2 + dy^2 + du^2 + dv^2 + \lambda^2(xdu - ydv + dz)^2,$$

$$\text{where} \quad \lambda > 0 \quad \text{arbitrary parameter.}$$

Typical symmetry of order 4 at the origin is given by

$$x' = -y, \; y' = x, \; z' = -z, \; u' = -v, \; v' = u.$$

Example 2. $M = SO(3,)/SO(2)$ with the invariant Riemannian metrics depending on 3 real parameters. The symmetry at the origin of M is as follows

$$\left\| \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right\| \mapsto \left\| \begin{array}{ccc} \bar{b}_2 & -\bar{b}_1 & \bar{b}_3 \\ -\bar{a}_2 & \bar{a}_1 & -\bar{a}_3 \\ \bar{c}_2 & -\bar{c}_1 & \bar{c}_3 \end{array} \right\|.$$

Example 3. Solvable complex matrix group

$$M = \left\| \begin{array}{ccc} e^{\lambda t} & 0 & z \\ 0 & e^{-\lambda t} & w \\ 0 & 0 & 1 \end{array} \right\| \cong \mathbb{R}^2[z, w] \times [t]$$

where z, w: complex variables, t: real variable, λ: complex parameter, $\lambda \neq 0$.

A family of invariant metrics g such that (M, g) is irreducible (i.e., not a product of Riemannian manifolds).

Symmetry at the origin is given by

$$z' = iw, \quad w' = iz, \quad t' = -t.$$

Example 4. $M = SO(3) \times SO(3)/SO(2)$

$$SO(2) = \left\{ \left(\left\| \begin{array}{ccc} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{array} \right\|, \left\| \begin{array}{ccc} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{array} \right\| \right), t \in \mathbb{R} \right\}$$

Symmetry of order 4 at the origin is induced by the following automorphisms of the group $GL(3,) \times GL(3,)$:

$$\left(\left\|\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right\|, \left\|\begin{array}{ccc} \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 \\ \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \\ \tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 \end{array}\right\|\right) \mapsto \left(\left\|\begin{array}{ccc} \tilde{a}_1 & -\tilde{a}_2 & -\tilde{a}_3 \\ -\tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \\ -\tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 \end{array}\right\|, \left\|\begin{array}{ccc} a_1 & -a_2 & a_3 \\ -b_1 & b_2 & -b_3 \\ c_1 & -c_2 & c_3 \end{array}\right\|\right)$$

Example 5. $M = SU(3)/SU(2) \cong S^5$ ((M, g) is not a standard sphere). For a special choice of the invariant metric, S^5 has the metric induced from the Fubini–Study metric in P^3 as a geodesic sphere in P^3.

4.3.2 A family of spaces of order 6

$$M = \left\|\begin{array}{cccc} e^{-(u+v)} & 0 & 0 & x \\ 0 & e^u & 0 & y \\ 0 & 0 & e^v & z \\ 0 & 0 & 0 & 1 \end{array}\right\| \cong \mathbb{R}^5[x, y, z, u, v] \qquad \text{(A solvable Lie group)}$$

We consider invarinat Riemannian metrics depending on two real parameters $a > 0, b > 0$:

$$\begin{aligned} g &= a^2(du^2 + dv^2 + dudv) + (b^2+1)(e^{2(u+v)}dx^2 + e^{-2u}dy^2 + e^{-2v}dz^2) \\ &\quad +(b^2-2)(e^v dxdy + e^u dxdz - e^{-(u+v)}dydz). \end{aligned}$$

The typical symmetry at the origin is given by

$$x' = y,\ y' = -z,\ z' = x,\ u' = v,\ v' = -(u+v).$$

This symmetry is of order 6!

The isotropy group of isometries at the origin consists of 8 elements.

Remark. Let (M, g) be not necessarily homogeneous in general. Let there exist a point $p \in M$ such that p is an isolated fixed point for some isometry $\exists s_p : M \longrightarrow M$. Then there exists an isometry $s'_p : M \longrightarrow M$ of finite order k:

$$k \leq (4 + s(p))^{\frac{\dim M + 1}{2}}$$

where $s(p)$ denotes the Singer number defined as follows: Put

$$G_p^{(l)} = \{A \in \mathrm{Aut}(T_pM) \mid A(g_p) = g_p, A(R_p) = R_p, \cdots, A((\nabla^{(l)}R)_p) = (\nabla^{(l)}R)_p\},$$

$l = 0, 1, 2, \cdots$. Then there exists the minimal $k \geq 0$ such that the sequence $\{G_p^{(l)}\}$ stabilizes, i.e.,

$$G_p^{(0)} \supset G_p^{(1)} \supset \cdots \supset G_p^{(k)} = G_p^{(k+1)} = \cdots$$

Such a number $k = s(p)$ is called the **Singer number**.

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COHOMOLOGY OF FLAT CONNECTIONS

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April 26, 2001

Abstract

We study flat connections in spherical Lie algebroids (over oriented compact manifolds) defined everywhere but a finite number of points. Under some assumptions concerning dimensions with any such isolated singularity we join a real number called an index. For $\mathbb{R}$-spherical Lie algebroids, this index cannot be an integer. We prove the index theorem saying that the index sum is independent of the choice of a connection. Multiplying this index sum by the orientation class of M, we get the Euler class of this Lie algebroid. Some integral formulae for indexes are given.

1 Introduction

Lie algebroids arise in many subjects of differential geometry and play a role analogous to that of Lie algebras for Lie groups (i.e. compose an infinitesimal invariant). For example, they arise in the theory of differential groupoids, principal bundles and vector bundles [P], [L2], [K-S], [M1], [K2], [N1], [N2], transversally complete and transversally parallelizable foliations [MO1], [MO2], nonclosed Lie subgroups [MO2], [K4], Poisson manifolds [C-D-W], [D-S], [G1], [V1], [V2], [Ko1], [Ko2] and others (see the survey article by K.Mackenzie [M2], and also see J.Kubarski [K6]). Connections – splittings of the Atiyah sequences of Lie algebroids (in regular case) – correspond in the majority of the above categories to well known geometric objects such as distributions or differential systems.

On the ground of Lie algebroids, we observe an interesting analogy of the theory of sphere bundles, namely, it turns out that, in some sense, the roles of flat connections for Lie algebroids and of cross-sections for sphere bundles correspond mutually. The common ideas are the index at a singularity and the theorem of Euler-Poincaré-Hopf type as well as some technique methods. This analogy was first noticed for Lie algebroids with 1-dimensional isotropy Lie algebras in the geometry of regular Poisson manifolds over $\mathbb{R}$-Lie foliations [K8]. The main purpose of our work is to research this phenomenon in the domain of transitive Lie algebroids without a restriction of the dimensions of isotropy Lie algebras.

This paper is based on [K7] and [K10]. In [K7] the idea of the fibre integral is adopted to regular Lie algebroids: the integration operator over the adjoint bundle of Lie algebras is defined, the class of Lie algebroids for which this operator commutes with differentials (giving then a homomorphism on cohomology) is characterized and many families of examples coming from principal bundles, TC-foliations and Poisson manifolds are given. Paper [K10] deals with a subclass of the above class which contains so-called *s-Lie algebroids,* defined as the transitive ones with spherical isotropy Lie algebras i.e. cohomologically looking like a sphere (Lie algebras $\mathbb{R}$, $sk\,(3,\mathbb{R})$, $sl\,(2,\mathbb{R})$ are examples). For an s-Lie algebroid there is constructed a long exact sequence of cohomology (Gysin types) and the *Euler class.*

In this paper the cohomology theory of flat connections in s-Lie algebroids is developed. If the dimension of the base manifold is equal to $n+1$, where n is the dimension of the isotropy Lie algebras, the index at a isolated singularity is defined. A version of the Euler-Poincaré-Hopf theorem joining the sum of indexes to the Euler class is given. In the context of Lie algebroids coming from S^1 or $Spin(3)$ principal bundles (over M^2 or M^4, respectively) the above theorem generalizes the classical E-P-H theorem since cross-sections of these bundles determine flat connections (but not every connection determines a cross-section). In the end, some integral formulae for indexes are obtained. We add that in general the index can not be an integer and the sum of indexes has nothing in common with the Euler-Poincaré characteristic of the Lie algebroid understood as an alternative sum of the dimensions of the cohomology groups of this Lie algebroid.

2 Fibre integral and Gysin sequence

By a *Lie algebroid on a manifold M* [P] we mean a system $A = (A, [\![\cdot,\cdot]\!], \gamma)$ consisting of a vector bundle A on M and mappings $[\![\cdot,\cdot]\!] : SecA \times SecA \to SecA$, $\gamma : A \to TM$, such that (1) $(SecA, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra, (2) γ, called the *anchor*, is a homomorphism of vector bundles, (3) $Sec\gamma : SecA \to \mathfrak{X}(M)$, $\xi \mapsto \gamma \circ \xi$, is a homomorphism of Lie algebras, (4) $[\![\xi, f \cdot \eta]\!] = f \cdot [\![\xi, \eta]\!] + (\gamma \circ \xi)(f) \cdot \eta$, $\xi, \eta \in SecA$, $f \in C^\infty(M)$. The properties make the space of global cross-sections $SecA$ a Lie-Rinehart algebra [called also a Lie module, a Lie pseudoalgebra, etc, according to particular authors] over the commutative algebra $C^\infty(M)$ [H1]. A Lie algebroid A is said to be *transitive* if γ is an epimorphism of vector bundles, and *regular* if γ is of constant rank. When the isotropy Lie algebras $\mathfrak{g}_{|x}$ of A are isomorphic to a given Lie algebra $\mathfrak{g}$ then A is shortly called a $\mathfrak{g}$-Lie algebroid. In the sequel, the notions and the notations from [P], [M1], [K3], [M-R], [K1], [M1] are adopted. Among them, the adjoint bundle of Lie algebras $\boldsymbol{g} := Ker\gamma$, the Atiyah sequence $0 \to \boldsymbol{g} \to A \xrightarrow{\gamma} F \to 0$, the notions of representations, connections, homomorphisms (strong or non-strong) of Lie algebroids as well as the exterior derivative d_A and the pullback of A-differential forms, are used.

In [K7] we introduce the notion of a *vertically oriented Lie algebroid* as a pair (A, ε) consisting of a regular Lie agebroid A and a non-singular cross-section ε of $\bigwedge^n \boldsymbol{g}$, $n = rank\boldsymbol{g}$. By a *homomorphism of vertically oriented Lie algebroids* $(A, \varepsilon) \to (A', \varepsilon')$, assuming $rank\boldsymbol{g} = rank\boldsymbol{g}' = n$, we mean a non-strong, in general, homomorphism $T : A \to A'$, inducing $t : M \to M'$, of Lie algebroids, such that $(\bigwedge^n T^+)(\varepsilon_x) = \varepsilon'_{tx}$, $x \in M$ (where $T^+ : \boldsymbol{g} \to \boldsymbol{g}'$ is the restriction of T to adjoint bundles). We write $(T, t) : (A, \varepsilon) \to (A', \varepsilon')$.

An operator of the fibre integral $⨍_A$ in a *vertically oriented Lie algebroid* (A, ε) is introduced. For a transitive Lie algebroid (the case considered in this work), the fibre integral $⨍_A : \Omega^\star_A(M) \to \Omega^{\star - n}(M)$ is defined in the following way: $⨍_A \Phi = 0$ if $\deg \Phi < n$ and $\gamma^\star \left(⨍_A \Phi\right) = (-1)^{nk} \iota_\varepsilon \Phi$ if $\deg \Phi = n + k$, $k \geqslant 0$. We recall that $\Omega_A(M)$ denotes the space of real A-differential forms, i.e. the space $Sec \bigwedge A^\star$ of cross-sections of the bundle $\bigwedge A^\star$. The following are basic properties (a) $⨍_A \circ \gamma^\star = 0$, (b) $⨍_A \gamma^\star \psi \wedge \Phi = \psi \wedge ⨍_A \Phi$ for arbitrary forms $\psi \in \Omega(M)$ and $\Phi \in \Omega_A(M)$, (c) $⨍_A$ is an epimorphism [K7].

The operator $⨍_A$ commutes with the exterior derivatives d_A and d_M if and only if (a1) the isotropy Lie algebras $\boldsymbol{g}_{|x}$ are unimodular, and (a2) the cross-section ε is invariant with respect to the adjoint representation of A on $\bigwedge^n \boldsymbol{g}$. The transitive Lie algebroid A fulfilling properties (a1) and (a2) above is shortly called a *TUIO-Lie algebroid.* In [K7] and [K9] many examples can be found. The paper [K10] deal with the subcategory of TUIO-Lie algebroids for which isotropy Lie algebras $\mathfrak{g}$ are spherical, i.e. satisfy conditions $H^k(\mathfrak{g}) = \mathbb{R}$ for $k = 0, \dim \mathfrak{g}$, and $H^k(\mathfrak{g}) = 0$ for $1 \leq k \leq \dim \mathfrak{g} - 1$. Such Lie algebroids are called briefly *s-Lie algebroids.* Many examples for principal bundles and TC-foliations are given. For an s-Lie algebroid (A, ε), there is constructed a long exact sequence of cohomology

(Gysin sequence)

$$\cdots \longrightarrow H^p(M) \xrightarrow{D^p} H^{p+n+1}(M) \xrightarrow{\gamma^\#} H_A^{p+n+1}(M) \xrightarrow{\int_A^\#} H^{p+1}(M) \xrightarrow{D^{p+1}} \cdots \tag{2.2.1}$$

The class $\chi_{(A,\varepsilon)} := D^0(1) \in H^{n+1}(M)$ is called the *Euler class* of (A,ε) and $\chi_{(A,\varepsilon)} = [\Psi]$ where $\gamma^*\Psi = d_A\Phi$ for $\int_A \Phi = -1$. The equality $D(\alpha) = \alpha \wedge \chi_{(A,\varepsilon)}$ holds. The Euler class $\chi_{(A,\varepsilon)}$ can be computed via the Chern-Weil homomorphism of A (introduced in [K3]).

3 Difference class

By a *connection* in a transitive Lie algebroid A we mean a splitting $\lambda : TM \to A$ of the Atiyah sequence $0 \to \boldsymbol{g} \hookrightarrow A \xrightarrow{\gamma} TM \to 0$, i.e. a homomorphism of vector bundles such that $\gamma \circ \lambda = id$. If λ is a homomorphism of Lie algebroids $\lambda \circ [X,Y] = [\![\lambda \circ X, \lambda \circ Y]\!]$, $X, Y \in \mathfrak{X}(M)$, then λ is called *flat.* In this situation, the pullback of differential forms $\lambda^\star : \Omega_A(M) \to \Omega(M)$ commutes with differentials $\lambda^\star \circ d_A = d_M \circ \lambda^\star$, giving — on cohomology — a homomorphism of algebras $\lambda^\# : H_A(M) \to H(M)$. Let λ be a flat connection in an s-Lie algebroid (A,ε). According to the exactness of sequence (2.2.1), [K7, Prop. 4.2.1 (b)] and [K10, Cor. 3.4] we can easily show that

$$H_A(M) = \ker \int_A^\# \bigoplus \ker \lambda^\#, \tag{3.3.1}$$

$$\int_A^\# \Big|_{\ker \lambda^\#} : \ker \lambda^\# \xrightarrow{\cong} H(M). \tag{3.3.2}$$

By the above, there exists a uniquely determined cohomology class

$$\omega_\lambda \in \ker\, \lambda^{\# n} \subset H_A^n(M)$$

such that $\int_A^\# \omega_\lambda = 1 \in H^0(M)$. ω_λ depends on the mapping $\lambda^\# : H_A(M) \to H(M)$ only. The class ω_λ is called the *cohomology class* of a flat connection λ.

For two flat connections $\lambda, \sigma : TM \to A$, their cohomology classes $\omega_\lambda, \omega_\sigma \in H_A^n(M)$ satisfy the equality $\int_A^\# (\omega_\lambda - \omega_\sigma) = 0$. Thanks to the exactness of sequence (2.2.1) there exists a cohomology class $[\lambda, \sigma] \in H^n(M)$ such that

$$\omega_\lambda - \omega_\sigma = \gamma^\# [\lambda, \sigma].$$

Definition 3.1. *The class* $[\lambda, \sigma]$ is called the *difference class* of flat connections λ and σ in the s-Lie algebroid (A,ε).

Proposition 3.2. *For flat connections λ and σ in an s-Lie algebroid (A,ε), we have* $\lambda^\#\alpha - \sigma^\#\alpha = -\left(\int_A^\# \alpha\right) \wedge [\lambda,\sigma], \quad \alpha \in H_A(M).$

Since $\gamma^\#$ is a monomorphism for a flat Lie algebroid A, we obtain

Corollary 3.3. *Let λ and σ be two flat connections in an s-Lie algebroid A. Then the following conditions are equivalent: (a) $\lambda^\# = \sigma^\#$, (b) $\omega_\lambda = \omega_\sigma$, (c) $[\lambda, \sigma] = 0$. If $H^n(M) = 0$, then, clearly, $H^n(M) \ni [\lambda,\sigma] = 0$, therefore $\lambda^\# = \sigma^\#$.*

Lemma 3.4. *Let (A,ε) be an arbitrary s-Lie algebroid with rank$\boldsymbol{g} = n$. For a representative Ψ of the Euler class $\chi_{(A,\varepsilon)}$ and an n-form $\Phi \in \Omega_A^n(M)$ such that $\int_A \Phi = -1$ and $d_A\Phi = \gamma^\star\Psi$, for any open subset $U \subset M$ and two flat connections $\lambda, \sigma : TU \to A_{|U}$, the following equalities hold:*

(1) $\omega_\sigma = [\gamma^\star_{|U}\sigma^\star\left(\Phi_{|U}\right) - \Phi_{|U}]$,

(2) $[\lambda,\sigma] = [\lambda^\star\Phi_{|U} - \sigma^\star\Phi_{|U}]$.

Theorem 3.5 (The naturality of the difference class). *Let* $(T,t) : (A,\varepsilon) \to (A',\varepsilon')$ *be a homomorphism of s-Lie algebroids such that* $T_x : A_{|x} \to A'_{|tx}$, $x \in M$, *are isomorphisms.*

- *(a) Assume that* σ,σ' *are flat connections in* A *and* A', *respectively, such that* $T\circ\sigma = \sigma'\circ t_*$. *Then* $T^\#\omega_{\sigma'} = \omega_\sigma$.
- *(b) For any two pairs of such flat connections* (λ,λ'), (σ,σ'), *we obtain* $t^\#([\lambda',\sigma']) = [\lambda,\sigma]$.

Proof. (a): To check (a), it is sufficient to notice that $\sigma^\#\left(T^\#\omega_{\sigma'}\right) = (T\circ\sigma)^\#\,\omega_{\sigma'} = (\sigma'\circ t_\star)^\#\,\omega_{\sigma'} = t^\#\sigma'^\#\omega_{\sigma'} = 0$ and (see [K7]) $\fint_A^\# T^\#\omega_{\sigma'} = t^\#\fint_{A'}^\#\omega_{\sigma'} = 1$.

(b): According to the fact that $\gamma^\#$ is a monomorphism, we only need to notice

$$\begin{aligned}\gamma^\# t^\#\left[\lambda',\sigma'\right] &= (t_\star\circ\gamma)^\#\left[\lambda',\sigma'\right] = \left(\gamma'\circ T\right)^\#\left([\lambda',\sigma']\right) = T^\#\left(\omega_{\lambda'} - \omega_{\sigma'}\right)\\ &= \omega_\lambda - \omega_\sigma = \gamma^\#\left[\lambda,\sigma\right].\end{aligned}$$

■

The folowing theorem gives a relationship between the Euler class and the difference class (compare with the classical theorem for sphere bundles, for example [1]).

Theorem 3.6. *Let* $\{U,V\}$ *be an open covering of* M *and let* $\lambda_U : TU \to A_{|U}$ *and* $\sigma_V : TV \to A_{|V}$ *be flat connections in* (A,ε) *over* U *and* V, *respectively* (U, V *need not be connected*). *Consider the Mayer-Vietoris sequence of the triad* $\{M,U,V\}$ *for the usual real de Rham cohomology and let* $\partial : H(U\cap V) \to H(M)$ *be the connecting homomorphism. Then*

$$\chi_{(A,\varepsilon)} = \partial\left[\lambda,\sigma\right]$$

where $\lambda = \lambda_U|_{U\cap V}$ *and* $\sigma = \sigma_V|_{U\cap V}$.

Proof. For the inclusions $j_1 : U\cap V \hookrightarrow U$ and $j_2 : U\cap V \hookrightarrow V$ according to Lemma 3.4, $[\lambda,\sigma] = \left[\lambda^\star\Phi_{|U\cap V} - \sigma^\star\Phi_{|U\cap V}\right] = [j_1^\star(\lambda_U^\star\Phi_{|U}) - j_2^\star(\sigma_V^\star\Phi_{|V})]$. Since $d(\lambda_U^\star\Phi_{|U}) = \lambda_U^\star d_{A_{|U}}\Phi_{|U} = \lambda_U^\star\gamma_{|U}^\star\Psi_{|U} = \Psi_{|U}$, analogously $d(\sigma_V^\star\Phi_{|V}) = \Psi_{|V}$, we get — via the construction of ∂ — $\partial\left[\lambda,\sigma\right] = [\Psi] = \chi_{(A,\varepsilon)}$. ■

4 The index of a flat connection at an isolated singular point and the Euler number

By a *local connection with singularity at a point* $a\in M$ in a Lie algebroid A we mean the connection

$$\sigma : T\dot U \to A_{|\dot U},\ a\in U\subset M\ \ (U \text{ is open}),\ \dot U = U\setminus\{a\}.$$

Let (A,ε) be an arbitrary s-Lie algebroid over an $n+1$-dimensional oriented manifold M ($n \geqslant 1$) with $n = rank\boldsymbol{g}$ and let $\sigma : T\dot U \to A_{|\dot U}$ be a local connection with singularity at $a \in U\subset M$. Take additionally a neighbourhood $V \ni a$ such that $V\subset U$ and $V \cong \mathbb{R}^{n+1}$. $A_{|V}$ possesses, [M1], a global flat connection $\lambda : TV \to A_{|V}$. Denote $\lambda|_{\dot V}$ ($\dot V = V\setminus\{a\}$) by $\dot\lambda$ and consider the difference class $[\dot\lambda,\sigma|_{\dot V}] \in H^n(\dot V)$. Let $\alpha_V : H^n(\dot V) \xrightarrow{\cong} \mathbb{R}$ be the canonical mapping [1, Vol.I] ($\dot{V}$ has the orientation induced from M). By analogous reasoning as in the theory of sphere bundles [1, Vol.I] and due to Corollary 3.3 we check that the number $\alpha_V([\dot{\lambda}, \sigma|_{\dot{V}}])$ is independent of the auxiliary flat connection λ and of the neighbourhood V. This means $\alpha_V([\dot{\lambda}, \sigma|_{\dot{V}}])$ depends only on the choice of σ.

Definition 4.1. The number $\alpha_V([\dot{\lambda}, \sigma|_{\dot{V}}])$ is called the *index of σ at a* and denoted by

$$j_a(\sigma).$$

Proposition 4.2 (Naturality of the index). *Let $(\hat{A}, \hat{\varepsilon})$ be another s-Lie algebroid over an oriented $n+1$-dimensional manifold $\hat{M}$ and $(T, t) : (\hat{A}, \hat{\varepsilon}) \to (A, \varepsilon)$ be a homomorphism of s-Lie algebroids fulfilling conditions*

$T_x : \hat{A}_{|x} \to A_{|tx},\ x \in M,$ *is an isomorphism,*
$t : \hat{M} \to M$ *is a diffeomorphism onto an open subset.*

Let $a \in M$, $\hat{a} \in \hat{M}$, $t(\hat{a}) = a$. Take a local flat connection $\sigma : T\dot{U} \to A_{|\dot{U}}$ with singularity at a. Then the mapping $T^{\#}\sigma : T\dot{W} \to \hat{A}_{|\dot{W}}$, $W = t^{-1}[U]$, $\dot{W} = W \setminus \{\hat{a}\}$, defined by $\left(T^{\#}\sigma\right)(v) = T_{|pv}^{-1}(\sigma(t_\star v))$ is a flat connection in $\hat{A}$ with singularity at $\hat{a}$, and $j_{\hat{a}}\left(T^{\#}\sigma\right) = j_a(\sigma)$.

The main goal of this article is a theorem joining the index sum $\sum j_{a_v}(\sigma)$ of any flat connection with a finite number of singularities $\{a_1, \ldots, a_k\}$ to the Euler class of Lie algebroid.

Theorem 4.3 (The Euler-Poincaré-Hopf theorem for flat connections). *Let (A, ε) be an s-Lie algebroid of rank n over an oriented compact manifold M of dimension $n+1$ and let $\sigma : T(M \setminus \{a_1, \ldots, a_k\}) \to A$ be a flat connection with singularities at points $a_1, \ldots, a_k$. Then the Euler class $\chi_{(A,\varepsilon)} \in H^{n+1}(M)$ is given by the formula*

$$\chi_{(A,\varepsilon)} = \left(\sum_{v=1}^{k} j_{a_v}(\sigma)\right) \cdot \omega_M$$

where ω_M is the orientation class of M; equivalently, $\int_M^{\#} \chi_{(A,\varepsilon)} = \sum_{v=1}^{k} j_{a_v}(\sigma)$. In particular, the index sum $\sum_{v=1}^{k} j_{a_v}(\sigma)$ is independent of the choice of the connection.

Proof. For each $v = 1, \ldots, k$, choose a neighbourhood $U_v \ni a_v$ diffeomorphic to $\mathbb{R}^{n+1}$ and such that the sets U_v are pairwise disjoint. Put $U = \bigcup U_v, \quad V = M \setminus \{a, \ldots, a_k\}$. Then $M = U \cup V$ and $U \cap V = \bigcup \dot{U}_v$ where $\dot{U}_v = U_v \setminus \{a_v\}$. Take arbitrary flat connections $\tilde{\lambda}_v : TU_v \to A_{|U_v}$, $v = 1, \ldots, k$. The family $\{\tilde{\lambda}_v\}$ determines one flat connection $\tilde{\lambda} : TU \to A_{|U}$ such that $\tilde{\lambda}|_{U_v} = \tilde{\lambda}_v$. Define $\check{\lambda} = \tilde{\lambda}|_{U \cap V}$ and $\check{\sigma} = \sigma|_{U \cap V}$. According to Theorem 3.6, $\chi_{(A,\varepsilon)} = \partial[\check{\lambda}, \check{\sigma}]$. In the sequel, put $\lambda_v = \tilde{\lambda}_v|_{\dot{U}_v}$ and $\sigma_v = \sigma|_{\dot{U}_v}$. Then $[\check{\lambda}, \check{\sigma}] = \oplus_v [\lambda_v, \sigma_v]$. By [1, Prop.VII Chap.VI, Vol.I] $\int_M^{\#} \circ \partial = \alpha$, where $\alpha : \bigoplus_v H^n(\dot{U}_v) \to \mathbb{R}$ is equal to $\oplus \beta_v \mapsto \sum \alpha_{U_v}(\beta_v)$. Therefore we get

$$\int_M^{\#} \chi_{(A,\varepsilon)A} = \int_M^{\#} \partial[\check{\lambda}, \check{\sigma}] = \int_M^{\#} \partial\left(\oplus_\nu [\lambda_v, \sigma_v]\right) = \alpha\left(\oplus_\nu [\lambda_v, \sigma_v]\right)$$
$$= \sum_{v=1}^{k} \alpha_{U_v}[\lambda_v, \sigma_v] = \sum_{v=1}^{k} j_{a_v}(\sigma).$$

■

The sum

$$j_{(A,\varepsilon)} = \sum_{v=1}^{k} j_{a_v}(\sigma)$$

is called the *Euler number of the s-Lie algebroid* (A, ε). According to Theorem 5.4 from [K10], the Euler number $j_{(A,\varepsilon)}$ is not – in general – an invariant of the cohomology algebra of A and has nothing in common with the Euler-Poincaré characteristic of A. The last, when considered for TUIO-Lie algebroids ($\dim M + rank\boldsymbol{g}$ is odd), always is 0 [K9].

5 Integral formulae

Proposition 5.1. *For a trivial s-Lie algebroid $A = TM \times \mathfrak{g}$ vertically oriented by a tensor $0 \neq \varepsilon_o \in \bigwedge^n \mathfrak{g}$ ($n = \dim \mathfrak{g}$) and equipped with a "standard" flat connection $\tau_0 : TM \to TM \times \mathfrak{g}$, $v \mapsto (v, 0)$, we have*

(a) *if $\sigma : TM \to A$ is a flat connection, then its cohomology class ω_σ is given by $\omega_\sigma = -\hat{\sigma}^{\#}[\varphi_o] \times 1 + 1 \times [\varphi_o]$ where $\hat{\sigma} = pr_2 \circ \sigma : TM \to \mathfrak{g}$ and $\varphi_o \in \bigwedge^n \mathfrak{g}^\star$ is a tensor such that $\iota_{\varepsilon_o}\varphi_o = 1$. In particular $\omega_{\tau_0} = 1 \times [\varphi_o]$.*

(b) *The difference class $[\tau_0, \sigma]$ is equal to $[\tau_0, \sigma] = \hat{\sigma}^{\#}[\varphi_o]$.*

Proof. Using the fact that the projection $pr_2 : TM \to \mathfrak{g}$ is a nonstrong homomorphism of Lie algebroids [K5], we can easily see that

$$\sigma^{\#}\left(-\hat{\sigma}^{\#}[\varphi_o] \times 1 + 1 \times [\varphi_o]\right) = 0,$$

$$\int_A^{\#}\left(-\hat{\sigma}^{\#}[\varphi_o] \times 1 + 1 \times [\varphi_o]\right) = 1.$$

Now (a) follows from the definition of the cohomology class ω_σ whereas (b) may be obtained from the definition of the difference class and the equality

$$\omega_{\tau_0} - \omega_\sigma = 1 \times [\varphi_o] - \left(-\hat{\sigma}^{\#}[\varphi_o] \times 1 + 1 \times [\varphi_o]\right) = \gamma^{\#}\hat{\sigma}^{\#}[\varphi_o].$$

■

Corollary 5.2. *For arbitrary flat connections $\lambda, \sigma : TM \to TM \times \mathfrak{g}$,*

$$[\lambda, \sigma] = (\hat{\sigma}^{\#} - \hat{\lambda}^{\#})[\varphi_o].$$

Put $\dot{\mathbb{R}}^{n+1} = \mathbb{R}^{n+1} \setminus \{0\}$ and let $\mathfrak{g}$ be any n-dimensional unimodular Lie algebra $\mathfrak{g}$. Take tensors $0 \neq \varepsilon_o \in \bigwedge^n \mathfrak{g}$, $\varphi_o \in \bigwedge^n \mathfrak{g}^\star$ joined by the relation $\iota_{\varepsilon_o}(\varphi_o) = 1$. Fix a flat connection $\sigma : T\dot{\mathbb{R}}^{n+1} \to T\dot{\mathbb{R}}^{n+1} \times \mathfrak{g}$ in the trivial Lie algebroid $A = T\mathbb{R}^{n+1} \times \mathfrak{g}$ (oriented by the tensor ε_o). Let $i : S^n \hookrightarrow \dot{\mathbb{R}}^{n+1}$ be the inclusion.

Proposition 5.3. *The index $j_0(\sigma)$ of σ is given by the formula*

$$j_0(\sigma) = \int_{S^n} \sigma_S^\star(\varphi_o) \tag{5.5.1}$$

where σ_S is a nonstrong homomorphism of Lie algebroids defined as the composition $\sigma_S : TS^n \xrightarrow{i_\star} T\dot{\mathbb{R}}^{n+1} \xrightarrow{\hat{\sigma}} \mathfrak{g}$.

Treat now $\hat{\sigma} : T\dot{\mathbb{R}}^{n+1} \to \mathfrak{g}$ as a 1-form on $\dot{\mathbb{R}}^{n+1}$ with values in $\mathfrak{g}$ and take the exterior n-product $\hat{\sigma} \wedge \ldots \wedge \hat{\sigma} \in \Omega^n(\dot{\mathbb{R}}^{n+1}; \bigwedge^n \mathfrak{g})$. We have $\hat{\sigma}^\star(\varphi_o) = \frac{1}{n!}(\varphi_o)_\star(\hat{\sigma} \wedge \ldots \wedge \hat{\sigma})$. Therefore

$$j_0(\sigma) = \frac{1}{n!}\varphi_o\left(\int_{S^n} i^\star(\hat{\sigma} \wedge \ldots \wedge \hat{\sigma})\right). \tag{5.5.2}$$

Example 5.4. Each trivial Lie algebroid $A = T\mathbb{R}^{n+1} \times \mathfrak{g}$ is integrable: $A = A(P)$ for $P = \mathbb{R}^{n+1} \times G$ where G is an arbitrary Lie group with the Lie algebra $\mathfrak{g}$. A connection $\sigma : T\dot{\mathbb{R}}^{n+1} \to A$ induces a connection $H \subset T(\dot{\mathbb{R}}^{n+1} \times G)$ in the principal bundle $\dot{\mathbb{R}}^{n+1} \times G$, and the flatness of σ means the integrability of H. Assume a leaf L of the foliation H is the graph of some function $f : \dot{\mathbb{R}}^{n+1} \to G$. (If $n \geq 2$, then such a function always exists which follows from the simple connectedness of $\dot{\mathbb{R}}^{n+1}$ and the reduction theorem [K-N]). Therefore $\hat{\sigma}(v) = R^{-1}_{f(x)\star}(f_\star v)$, $R_{f(x)}$ *which is the* right translation on G by $f(x)$, and $f^\star(\Delta_R) = \langle \varphi_o, \frac{1}{n!}(\hat{\sigma} \wedge \ldots \wedge \hat{\sigma})\rangle$ for the right-invariant n-form $\Delta_R \in \Omega^n_R(G)$ equalling φ_o at the unity e of G.

(A) If G is compact, n-dimensional, oriented by Δ_R and the Lie algebra of G is spherical, then as a consequence of (5.5.1) and (5.5.2) we have

$$j_0(\sigma) = \int_{S^n} \left(f_{|S^n}\right)^\star \Delta_R = deg\left(f_{|S^n}\right) \cdot \int_G \Delta_R\,. \tag{5.5.3}$$

As a corollary (taking any mapping $f : \dot{\mathbb{R}}^{n+1} \to S^n$ such that $f_{|S^n} = id_{S^n}$), we obtain the existence of a local, flat singular connection having a nonzero index at the singularity.

Formula (5.5.3) yields that the set of real numbers being the indexes at a given point of singular *local,* flat connections coming from functions is discrete (more exactly, is equal to the set of multiples of $\int_G \Delta_R$). Such a situation takes place, for example, for all flat connections in any $sk(3, \mathbb{R})$-Lie algebroid over M^4 (since we can take $G = SO(3)$).

(B) If G is not compact, then $\Delta_R = d(\Theta)$ for some Θ and

$$j_0(\sigma) = \int_{S^n} \left(f_{|S^n}\right)^\star \Delta_R = \int_{S^n} d\left(f^*_{|S^n}\Theta\right) = 0.$$

Such a situation takes place, for example, in any $sl(2, \mathbb{R})$-Lie algebroid over M^4 (since we can take $G = SL(2, \mathbb{R})$). Clearly, this fact can be noticed immediately by using a base e, f, g of $sl(2, \mathbb{R})$ such that $[e, f] = g$, $[f, g] = 2f$, $[g, e] = 2e$.

For an $\mathbb{R}$-Lie algebroid, not every local, singular and flat connection comes from a function, see the below example.

Example 5.5. In any $\mathbb{R}$-Lie algebroid over M^2 we can construct a local, flat and singular connection whose index is a preassigned real number. Indeed, since $\mathbb{R}$ is abelian, therefore the flatness of σ is equivalent to the closedness of the 1-form $\hat{\sigma}$ on M^2. In this case, the product $k \cdot \hat{\sigma}$, $k \in \mathbb{R}$, also gives a flat connection. Therefore, if $\sigma : T\dot{\mathbb{R}}^2 \to T\dot{\mathbb{R}}^2 \times \mathbb{R}$, $v \mapsto (v, \hat{\sigma}(v))$, is a flat connection with a nonzero index at 0, $j_0(\sigma) \neq 0$, then, for an arbitrary real number $k \in \mathbb{R}$, the mapping

$$\tau : T\dot{\mathbb{R}}^2 \longrightarrow T\dot{\mathbb{R}}^2 \times \mathbb{R}, \ \ v \longmapsto \left(v, \frac{k}{j_0(\sigma)} \cdot \hat{\sigma}(v)\right),$$

is a flat connection with $j_0(\tau) = k$. Except for a discrete set of real numbers, this connection does not come from a function. More implicitly, considering $\varepsilon_0 = 1 \in \mathbb{R}$ and taking $\hat{\tau} = \frac{k}{2\pi}\left(\frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx\right)$, $k \in \mathbb{R}$, we have $j_0(\tau) = \int_{S^1} \hat{\tau} = k$.

Example 5.6. In the Hopf S^1-bundle $P = (S^3 \to S^2)$, for two different points $p_1, p_2 \in S^2$ and for any real number $k \in \mathbb{R}$, there exists a global flat connection σ_k with two singularities at $\{p_1, p_2\}$, such that the index $j_{p_1}(\sigma_k)$ is equal to k. Indeed, since the Euler class of P is equal to the orientation class of S^2, any flat connection λ with a singularity at $\{p_1\}$ has the index at p_1 equal to 1 (assuming $\int_G \Delta_R = 1$). Take $p_2 \neq p_1$ and $M = S^2 \setminus \{p_2\}$. Since M is contractible, $P_{|M}$ is trivial $P_{|M} \cong M \times S^1$. The connection λ determines a connection $\bar{\lambda} : T(M \setminus \{p_1\}) \to A\left((M \setminus \{p_1\}) \times S^1\right) = T(M \setminus \{p_1\}) \times \mathbb{R}$. Take $\hat{\sigma} = pr_2 \circ \bar{\lambda} : T(M \setminus \{p_1\}) \to \mathbb{R}$. For an arbitrary real number $k \in \mathbb{R}$,

$$\bar{\sigma}_k : T(M \setminus \{p_1\}) \longrightarrow T(M \setminus \{p_1\}) \times \mathbb{R}, \quad v \longmapsto (v, k \cdot \hat{\sigma}(v)),$$

is a flat connection. $\bar{\sigma}_k$ determines a flat connection σ_k in P with a singularity at $\{p_1, p_2\}$, such that $j_{p_1}(\sigma_k) = k$.

In the end we give some remarks concerning the existence of a connection with a finite number of singularities. We start with the case $\mathfrak{g} = \mathbb{R}$.

Proposition 5.7. *In each invariantly oriented $\mathbb{R}$-Lie algebroid over an arbitrary manifold M for which $H^2(M) = 0$ there exists a flat connection, in particular, when M is 2-dimensional non-compact.*

Proof. According to [K7], each invariantly oriented $\mathbb{R}$-Lie algebroid A over M is isomorphic to $(M \times \mathbb{R}) \bigoplus TM$ with $pr_2 : (M \times \mathbb{R}) \bigoplus TM \to TM$ as the anchor and the bracket $[\![\cdot,\cdot]\!]$ defined via some closed real 2-form Ω in the following way: $[\![(f,X),(g,Y)]\!] = (-\Omega(X,Y) + \partial_X g - \partial_Y f, [X,Y]), X, Y \in \mathfrak{X}(M), f, g \in C^\infty(M)$. Each connection $\lambda : TM \to (M \times \mathbb{R}) \bigoplus TM$ has the form $\lambda(v) = \left(\bar{\lambda}(v), v\right)$ for a 1-differential form $\bar{\lambda} \in \Omega^1(M)$. A simple calculation shows that λ is flat if and only if $d\left(\bar{\lambda}\right) = \Omega$. If $H^2(M) = 0$, such a 1-form exists. ■

As a corollary we get

Corollary 5.8. *In each s-Lie algebroid of rank 1 over a compact 2-manifold M there exists a flat connection with a beforehand finite non-empty set of isolated singularities.*

If a $sk(3,R)$-Lie algebroid over a compact 4-manifold comes from a $Spin(3)$-principal bundle, then - of course - it possesses a flat connection with one singularity (since such a cross-section of the sphere bundle exists [1, Vol. I]). In the general case, the problem is open.

The problem for $sl(2,\mathbb{R})$-Lie algebroids looks differently. Namely, by the main theorem (4.3) and Example 5.4 (B) we have that $\chi_{(A,\varepsilon)} = 0$ for any invariantly oriented $sl(2,\mathbb{R})$-Lie algebroid (A,ε) over a compact connected oriented manifold M^4, admitting a flat connection with a finite number of isolated singularities. Really, such a Lie algebroid is flat since locally we can remove a singularity: if σ is a flat connection in $T\dot{\mathbb{R}}^4 \times sl(2,\mathbb{R})$ then σ is given by a function $f : \dot{\mathbb{R}}^4 \to SL(2,\mathbb{R})$. Using the fact that the third group of homotopy of $SL(2,\mathbb{R})$ is zero, $\pi_3(SL(2,\mathbb{R})) = 0$, we can find $\bar{f} : \mathbb{R}^4 \to SL(2,\mathbb{R})$ such that $f(x) = \bar{f}(x)$ for $\|x\| \geq \varepsilon$ for a given small ε. This implies that we may remove the singularity at 0.

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CONFIGURATION SPACES AND ALGEBROIDS

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Abstract

Let M be complete, connected, oriented, C^∞ (noncompact) Riemannian manifold of dimension d. The configuration space Γ_M over M is the set of locally finite subsets in M:

$$\Gamma_M := \{\gamma \subset M : \ \mathrm{card}\,(\gamma \cap K) < \infty \text{ for each compact } K \subset M\}.$$

Any $\gamma \in \Gamma_M$ is identified with positive integer-valued Radon measure. The tangent space to Γ_M at a point γ is defined as Hilbert space $T_\gamma\Gamma_M := L^2(M \to TM; \gamma)$ (or, equivalently, $T_\gamma\Gamma_M := \bigoplus_{x\in\gamma} T_x M$). (See S.Albeverio, Yu.G.Kondratiev, M.Röckner, JFA 157 (1998).) High order differential forms and de Rham cohomology on configuration spaces can also be considered (see S.Albeverio, A.Daletskii, E.Lytvynov, JGeomPhys, to appear). The main topic of discussion is to consider these objects from the Lie algebriods theory point of view.

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EXTENDED FINITE CALCULUS – AN EXAMPLE OF ALGEGRAIZATION OF ANALYSIS

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Abstract

"A Calculus of Sequences" started in 1936 by Ward constitutes the general scheme for extensions of classical operator calculus of Rota - Mullin considered by many afterwards and after Ward. Because of the notation we shall call the Wards calculus of sequences in its afterwards elaborated form - a ψ-calculus.

The ψ-calculus in parts appears to be almost automatic, natural extension of classical operator calculus of Rota - Mullin or equivalently - of umbral calculus of Roman and Rota.

At the same time this calculus is an example of the algebraization of the analysis - here restricted to the algebra of polynomials. Many of the results of ψ-calculus may be extended to Markowsky Q-umbral calculus where Q stands for a generalized difference operator i.e. the one lowering the degree of any polynomial by one.

The article is supplemented by the short indicatory glossaries of terms and notation used by Ward, Viskov, Markowsky, Roman on one side and the Rota-oriented notation on the other side.

KEY WORDS: extended umbral calculus , Graves-Heisenberg-Weyl algebra
MCS (2000) : 05A40 , 81S99

1 Introduction

We shall call the Wards calculus of sequences [1] in its afterwards last century elaborated form - a ψ-calculus because of the notation [2]-[7]. The efficiency of the Rota oriented language and our notation used has been already exemplified by easy proving of ψ-extended counterparts of all representation independent statements of ψ-calculus [2]. Here these are ψ-labelled representations of Graves-Heisenberg-Weyl (GHW) algebra of linear operators acting on the algebra P of polynomials.

As a matter of fact ψ-calculus becomes in parts almost automatic extension of Rota - Mullin calculus or equivalently - of umbral calculus of Roman and Rota [8, 9, 10]. The ψ-extension relies on the notion of ∂_ψ-shift invariance of operators with ψ-derivatives ∂_ψ staying for equivalence classes representatives of special differential operators lowering degree of polynomials by one [6, 7, 11]. Many of the results of ψ-calculus may be extended to Markowsky Q-umbral calculus [11] where Q stands for arbitrary generalized difference operator i.e. the one lowering the degree of any polynomial by one. Q-umbral calculus [11] - as we call it - includes also those generalized difference operators, which are not series in ψ-derivative ∂_ψ whatever an admissible ψ sequence would be.

The note is at the same time the operator formulation of "A Calculus of Sequences" started in 1936 by Ward [1] with the indication of the role the ψ-representations of Graves-Heisenberg-Weyl (GHW) algebra in formulation and derivation of principal statements of the ψ-extension of finite operator calculus of Rota.

Restating what was said above we observe that all statements of standard finite operator calculus of Rota are valid also in the case of ψ-extension under the almost automatic replacement of $\{D, \hat{x}, id\}$ generators of GHW by their ψ-representation correspondents $\{\partial_\psi, \hat{x}_\psi, id\}$ - see definitions 2.1 and

2.5. Naturally any specification of admissible ψ - for example the famous one defining q-calculus - has its own characteristic properties not pertaining to the standard case of Rota calculus realisation. Nevertheless the overall picture and system of statements depending only on GHW algebra is the same modulo some automatic replacements in formulas demonstrated in the sequel. The large part of that kind of job was already done in [2, 3].

The aim of this presentation is to give a general picture of the algebra of linear operators on polynomial algebra. The picture that emerges discloses the fact that any ψ-representation of finite operator calculus or equivalently - any ψ-representation of GHW algebra makes up an example of the algebraization of the analysis - naturally when constrained to the algebra of polynomials.

We shall delimit all our considerations to the algebra P of polynomials or sometimes to the algebra of formal series. Therefore the distinction in-between difference and differentiation operators disappears. All linear operators on P are both difference and differentiation operators if the degree of differentiation or difference operator is unlimited.

If all this is extended to Markowsky Q-umbral calculus then many of the results of ψ-calculus may be extended to Q-umbral calculus [11]. This is achieved under the almost automatic replacement of $\{D, \hat{x}, id\}$ generators of GHW or their ψ-representation $\{\partial_\psi, \hat{x}_\psi, id\}$ by their Q-representation correspondents $\{Q, \hat{x}_Q, id\}$ - see definition 2.5.

The article is supplemented by the short indicatory glossaries of terms and notation used by Ward, Viskov, Markowsky, Roman on one side and the Rota-oriented notation on the other side.

2 Primary definitions, notation and general observations

In the following we shall consider the algebra P of polynomials $P = \mathbf{F}[x]$ over the field $\mathbf{F}$ of characteristic zero. All operators or functionals studied here are to be understood as *linear* operators on P. It shall be easy to see that they are always well defined.

Throughout the note while saying "polynomial sequence $\{p_n\}_o^\infty$" we mean deg $p_n = n$; $n \geq 0$ and we adopt also the convention that deg $p_n < 0$ iff $p \equiv 0$.

Consider $\Im$ - the family of functions‘ sequences (in conformity with Viskov notation) such that: $\Im = \{\psi; R \supset [a,b]\ ;\ q \in [a,b]\ ;\ \psi(q) : Z \to F\ ;\ \psi_0(q) = 1\ ;\ \psi_n(q) \neq 0;\ \psi_{-n}(q) = 0;\ n \in N\}$.
We shall call $\psi = \{\psi_n(q)\}_{n\geq 0}$; $\psi_n(q) \neq 0$; $n \geq 0$ and $\psi_0(q) = 1$ an admissible sequence. Let now n_ψ denotes [2, 3]

$$n_\psi \equiv \psi_{n-1}(q)\,\psi_n^{-1}(q).$$

Then

$$n_\psi! \equiv \psi_n^{-1}(q) \equiv n_\psi (n-1)_\psi (n-2)_\psi (n-3)_\psi \ldots .2_\psi 1_\psi; \quad 0_\psi! = 1$$

$$n_\psi^{\underline{k}} = n_\psi (n-1)_\psi \ldots (n-k+1)_\psi \text{ and } \binom{n}{k}_\psi \equiv \frac{n_\psi^{\underline{k}}}{k_\psi!} \text{ and } \boldsymbol{exp}_\psi\{y\} = \sum_{k=0}^{\infty} \frac{y^k}{k_\psi!}.$$

Definition 2.1. *Let ψ be admissible. Let ∂_ψ be the linear operator lowering degree of polynomials by one defined according to $\partial_\psi x^n = n_\psi x^{n-1}$; $n \geq 0$. Then ∂_ψ is called the ψ-derivative.*

Remark 2.1. The choice $\psi_n(q)=[R(q^n)!]^{-1}$ and $R(x) = \frac{1-x}{1-q}$ results in the well known q-factorial $n_q! = n_q (n-1)_q!$; $1_q! = 0_q! = 1$ while the ψ-derivative ∂_ψ becomes now $(n_\psi = n_q)$ the Jackson's derivative [2, 3] ∂_q:

$(\partial_q \varphi)(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)x}$.

Note also that that if $\psi = \{\psi_n(q)\}_{n\geq 0}$ and $\varphi = \{\varphi_n(q)\}_{n\geq 0}$ are two admissible sequences then $[\partial_\psi\ ,\ \partial_\varphi]= 0$ iff $\psi = \varphi$.

Definition 2.2. *Let $E^y(\partial_\psi) \equiv exp_\psi\{y\partial_\psi\} = \sum_{k=0}^{\infty} \frac{y^k \partial_\psi^k}{k_\psi!}$. $E^y(\partial_\psi)$ is called the generalized translation operator.*

Note 2.1. [2, 3]
$E^a(\partial_\psi) f(x) \equiv f(x +_\psi a)$; $(x +_\psi a)^n \equiv E^a(\partial_\psi) x^n$; $E^a(\partial_\psi) f = \sum_{n\geq 0} \frac{a^n}{n_\psi!} \partial_\psi^n f$;
and in general $(x +_\psi a)^n \neq (x +_\psi a)^{n-1}(x +_\psi a)$.
Note also that in general $(1 +_\psi (-1))^{2n+1} \neq 0$; $n \geq 0$ though $(1 +_\psi (-1))^{2n} = 0$; $n \geq 1$.

Note 2.2. [1]
$\exp_\psi(x +_\psi y) \equiv exp_\psi\{x\} \exp_\psi\{y\}$ - while in general $\exp_\psi\{x + y\} \neq \exp_\psi\{x\} \exp_\psi\{y\}$.

Possible consequent utilisation of the identity $\exp_\psi(x +_\psi y) \equiv \exp_\psi\{x\} \exp_\psi\{y\}$ is quite encouraging. It leads among others to "ψ-trigonometry" either ψ-elliptic or ψ-hyperbolic via introducing $\cos_\psi$, $\sin_\psi$ [1], $\cosh_\psi$, $\sinh_\psi$ or in general ψ-hyperbolic functions of m-th order $\left\{h_j^{(\psi)}(\alpha)\right\}_{j\in Z_m}$ defined according to [12]

$$R \ni \alpha \rightarrow\ h_j(\alpha) = \frac{1}{m} \sum_{k\in Z_m} \omega^{-kj} \exp_\psi\left(\omega^k \alpha\right)\ ;\ j \in Z_m,\ \omega = \exp\left(i\frac{2\pi}{m}\right).$$

where $1 < m \in N$ and $Z_m = \{0, 1, ..., m-1\}$.

Definition 2.3. *A polynomial sequence $\{p_n\}_o^\infty$ is of ψ-binomial type if it satisfies the recurrence*

$$E^y(\partial_\psi) p_n(x) \equiv p_n(x +_\psi y) \equiv \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x) p_{n-k}(y).$$

Polynomial sequences of ψ-binomial type [2, 3] are known to correspond in one-to-one manner to special generalized differential operators Q, namely to those $Q = Q(\partial_\psi)$ which are ∂_ψ-shift invariant operators [2, 3]. We shall deal in this note mostly with this special case i.e. with ψ-umbral calculus. However before to proceed let us supply a basic information referring to this general case of Q-umbral calculus.

Definition 2.4. *Let $P = \mathbf{F}[x]$. Let Q be a linear map $Q\ :\ P \rightarrow P$ such that: $\forall_{p\in P}$ $deg\ (Qp) = (deg\ p) - 1$ (with the convention $deg\ p = -1$ means $p = const = 0$). Q is then called a generalized difference-tial operator [11] or Gel'fond-Leontiev [7] operator.*

Right from the above definitions we infer that the following holds.

Observation 2.1. Let Q be as in Definition 2.4. Let $Qx^n = \sum_{k=1}^{n} b_{n,k}x^{n-k}$ where $b_{n,1} \neq 0$ of course. Without loose of generality take $b_{1,1} = 1$. Then $\exists\ \{q_k\}_{q\geq 2} \subset \mathbf{F}$ and $\exists$ admissible ψ such that

$$Q = \partial_\psi + \sum_{k\geq 2} q_k \partial_\psi^k \tag{2.2.1}$$

if and only if

$$b_{n,k} = \binom{n}{k}_\psi b_{k,k};\quad n \geq k \geq 1;\ b_{n,1} \neq 0;\ b_{1,1} = 1. \tag{2.2.2}$$

If $\{q_k\}_{q\geq 2}$ and an admissible ψ exist then these are unique.

Notation 2.1. In the case (2.2.2) is true we shall write : $Q = Q(\partial_\psi)$.

Remark 2.2. Note that operators of the (2.2.1) form constitute a group under superposition of formal power series (compare with the formula (S) in [13]). Of course not all generalized difference-tial operators satisfy (2.2.1) i.e. are series just only in corresponding ψ-derivative ∂_ψ (see Proposition 3.1). For example [14] let $Q = \frac{1}{2}D\hat{x}D - \frac{1}{3}D^3$. Then $Qx^n = \frac{1}{2}n^2x^{n-1} - \frac{1}{3}n^{\underline{3}}x^{n-3}$ so according to Observation 2.1 $n_\psi = \frac{1}{2}n^2$ and there exists no admissible ψ such that $Q = Q(\partial_\psi)$.

Observation 2.2. From theorem 3.1 in [11] we infer that generalized differential operators give rise to subalgebras $\sum_Q$ of linear maps (plus zero map of course) commuting with a given generalized difference-tial operator Q. The intersection of two different algebras $\sum_{Q_1}$ and $\sum_{Q_2}$ is just zero map added.

The importance of the above Observation 2.2 as well as the definition below may be further fully appreciated in the context of the Theorem 2.1 and the Proposition 3.1 to come.

Definition 2.5. *Let $\{p_n\}_{n\geq 0}$ be the normal polynomial sequence [11] i.e. $p_0(x) = 1$ and $p_n(0) = 0$; $n \geq 1$. Then we call it the ψ-basic sequence of the generalized difference-tial operator Q if in addition $Qp_n = n_\psi p_{n-1}$. Parallely we define a linear map $\hat{x}_Q$: $P \rightarrow P$ such that $\hat{x}_Q p_n = \frac{(n+1)}{(n+1)_\psi} p_{n+1}$; $n \geq 0$. We call the operator $\hat{x}_Q$ the dual to Q operator.*

When $Q = Q(\partial_\psi) = \partial_\psi$ we write for short: $\hat{x}_{Q(\partial_\psi)} \equiv \hat{x}_{\partial_\psi} \equiv \hat{x}_\psi$ (see: Definition 2.9).
Of course $[Q, \hat{x}_Q] = id$ therefore $\{Q, \hat{x}_Q, id\}$ provide us with a continuous family of generators of GHW in - as we call it - Q-representation of Graves-Heisenberg-Weyl algebra.
In the following we shall restrict to special case of generalized differential operators Q, namely to those $Q = Q(\partial_\psi)$ which are ∂_ψ-shift invariant operators [2, 3] (see: Definition 2.6).

At first let us start with appropriate ψ-Leibnitz rules for corresponding ψ-derivatives.
ψ-Leibnitz rules:

It is easy to see that the following hold for any formal series f and g:
for ∂_q: $\partial_q(f \cdot g) = (\partial_q f) \cdot g + \left(\hat{Q}f\right) \cdot (\partial_q g)$, where $\left(\hat{Q}f\right)(x) = f(qx)$;
for $\partial_R = R\left(q\hat{Q}\right)\partial_o$: $\partial_R(f \bullet g)(z) = R\left(q\hat{Q}\right)\{(\partial_o f)(z) \bullet g(z) + f(0)(\partial_o g)(z)\}$
where - note - $R\left(q\hat{Q}\right)x^{n-1} = n_R x^{n-1}$; $(n_\psi = n_R = n_{R(q)} = R(q^n))$ and finally
for $\partial_\psi = \hat{n}_\psi \partial_o$:

$$\partial_\psi(f \bullet g)(z) = \hat{n}_\psi\{(\partial_o f)(z) \bullet g(z) + f(0)(\partial_o g)(z)\}$$

where $\hat{n}_\psi x^{n-1} = n_\psi x^{n-1}$; $n \geq 1$.

Example 2.1. Let $Q(\partial_\psi) = D\hat{x}D$, where $\hat{x}f(x) = xf(x)$ and $D = \frac{d}{dx}$. Then $\psi = \left\{\left[\left(n^2\right)!\right]^{-1}\right\}_{n\geq 0}$ and $Q = \partial_\psi$. Let $Q(\partial_\psi)R(q\hat{Q})\partial_0 \equiv \partial_R$. Then $\psi = \left\{\left[R(q^n)!\right]^{-1}\right\}_{n\geq 0}$ and $Q = \partial_\psi \equiv \partial_R$. Here $R(z)$ is any formal Laurent series; $\hat{Q}f(x) = f(qx)$ and $n_\psi = R(q^n)$. ∂_0 is $q = 0$ Jackson derivative which as a matter of fact - being a difference operator is the differential operator of infinite order at the same time:

$$\partial_o = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{n-1}}{n!}\frac{d^n}{dx^n}. \tag{2.2.3}$$

Naturally with the choice $\psi_n(q) = \left[R(q^n)!\right]^{-1}$ and $R(x) = \frac{1-x}{1-q}$ the ψ-derivative ∂_ψ becomes the Jackson's derivative [2, 3] ∂_q:

$$(\partial_q \varphi)(x) = \frac{1 - q\hat{Q}}{(1-q)}\partial_0 \varphi(x).$$

The equivalent to (2.2.3) form of Bernoulli-Taylor expansion one may find [15] in *Acta Eruditorum* from November 1694 under the name "*series univeralissima*".

(Taylor's expansion was presented in his "Methodus incrementorum directa et inversa" in 1715 - edited in London).

Definition 2.6. *Let us denote by $End(P)$ the algebra of all linear operators acting on the algebra P of polynomials. Let*

$$\sum\nolimits_{\psi} = \{T \in End(P);\ \forall\ \alpha \in F;\ [T; E^{\alpha}(\partial_{\psi})] = 0\}.$$

Then $\sum_{\psi}$ is a commutative subalgebra of $End(P)$ of F-linear operators. We shall call these operators T : ∂_{ψ}-shift invariant operators.

We are now in a position to define further basic objects of "ψ-umbral calculus" [2, 3].

Definition 2.7. *Let $Q(\partial_{\psi}) : P \to P$; the linear operator $Q(\partial_{\psi})$ is a ∂_{ψ}-delta operator iff*

1. *$Q(\partial_{\psi})$ is ∂_{ψ} - shift invariant;*
2. *$Q(\partial_{\psi})(id) = const \neq 0$*

The strictly related notion is that of the ∂_{ψ}-basic polynomial sequence:

Definition 2.8. *Let $Q(\partial_{\psi}) : P \to P$; be the ∂_{ψ}-delta operator. A polynomial sequence $\{p_n\}_{n\geq 0}$; deg $p_n = n$ such that:*

1. *$p_o(x) = 1$;*
2. *$p_n(0) = 0$; $n > 0$;*
3. *$Q(\partial_{\psi}) p_n = n_{\psi} p_{n-1}$ is called the ∂_{ψ}-basic polynomial sequence of the ∂_{ψ}-delta operator$Q(\partial\psi)$.*

Identification 2.1. It is easy to see that the following identification takes place: ∂_{ψ}-delta operator $Q(\partial_{\psi}) = \partial_{\psi}$-shift invariant generalized differential operator Q. Of course not every generalized differential operator might be considered to be such.

Note: Let $\Phi(x;\lambda) = \sum\limits_{n\geq 0} \frac{\lambda^n}{n_{\psi}!} p_n(x)$ denotes the ψ-exponential generating function of the ∂_{ψ}-basic polynomial sequence $\{p_n\}_{n\geq 0}$ of the ∂_{ψ}-delta operator $Q \equiv Q(\partial_{\psi})$ and let $\Phi(0;\lambda) = 1$. Then $Q\Phi(x;\lambda) = \lambda\Phi(x;\lambda)$ and Φ is the unique solution of this eigenvalue problem. In view of Observation 2.2 we affirm that then exists such an admissible sequence φ that $\Phi(x;\lambda) = \exp_{\varphi}[\lambda x]$.

The notation and naming established by Definitions 2.7 and 2.8 serve the target to preserve and to broaden simplicity of Rota's finite operator calculus also in its extended "ψ-umbral calculus" case [2, 3]. As a matter of illustration of such notation efficiency let us quote after [2] the important Theorem 2.1 which might be proved using the fact that $\forall\ Q(\partial_{\psi})\quad \exists!$ invertible $S \in \Sigma_{\psi}$ such that $Q(\partial_{\psi}) = \partial_{\psi} S$. (For Theorem 2.1 see also Theorem 4.3. in [11], which holds for operators, introduced by the Definition 2.5).

Theorem 2.1. (ψ-Lagrange and ψ-Rodrigues formulas)
Let $\{p_n(x)\}_{n=0}^{\infty}$ be ∂_{ψ}-basic polynomial sequence of the ∂_{ψ}-delta operator $Q(\partial_{\psi})$.
Let $Q(\partial_{\psi}) = \partial_{\psi} S_{\partial_{\psi}}$. Then for $n > 0$:

1. $p_n(x) = Q(\partial_{\psi})' \, S_{\partial_{\psi}}^{-n-1}\, \mathrm{x}^n$;
2. $p_n(x) = S_{\partial_{\psi}}^{-n} \mathrm{x}^n - \frac{n_{\psi}}{n} (S_{\partial_{\psi}}^{-n})' \mathrm{x}^{n-1}$;
3. $p_n(x) = \frac{n_{\psi}}{n} \hat{x}_{\psi} S_{\partial_{\psi}}^{-n} \mathrm{x}^{n-1}$;
4. $p_n(x) = \frac{n_{\psi}}{n} \hat{x}_{\psi} (Q(\partial_{\psi})')^{-1} p_{n-1}(x)$ ($\leftarrow$ Rodrigues ψ-formula).

For the proof one uses typical properties of the Pincherle ψ-derivative defined bellow as well as $\hat{x}_{\psi}$ operator.

Definition 2.9. (compare with (17) in [7])
The Pincherle ψ-derivative i.e. the linear map $' : \Sigma_\psi \quad \rightarrow \quad \Sigma_\psi$;
$T \ ' = T\, \hat{x}_\psi$ - $\hat{x}_\psi T \equiv [T_{\partial_\psi}\ , \hat{x}_\psi]$
where the linear map $\hat{x}_\psi : P \rightarrow P$; *is defined in the basis* $\{x^n\}_{n\geq 0}$ *as follows*

$$\hat{x}_\psi x^n = \frac{\psi_{n+1}(q)(n+1)}{\psi_n(q)} x^{n+1} = \frac{(n+1)}{(n+1)_\psi} x^{n+1}; \quad n \geq 0$$

Observation 2.3. [2,3]
The triples $\{\partial_\psi, \hat{x}_\psi, id\}$ for any admissible ψ-constitute the set of generators of the ψ-labelled representations of Graves-Heisenberg-Weyl (GHW) algebra [16, 17, 18]. Namely, as easily seen $[\partial_\psi, \hat{x}_\psi]$ $= id$. (compare with Definition 2.5)

Observation 2.4. In view of the Observation 2.3 the general Leibnitz rule in ψ-representation of Graves-Heisenberg-Weyl algebra may be written (compare with 2.2.2 Proposition in [17]) as follows

$$\partial_\psi^n \quad \hat{x}_\psi^m \quad = \quad \sum_{k\geq 0} \binom{n}{k} \binom{m}{k} k!\, \hat{x}_\psi^{m-k}\, \partial_\psi^{n-k}. \tag{2.2.4}$$

One derives the above ψ-Leibnitz rule from ψ-Heisenberg-Weyl exponential commutation rules exactly the same way as in $\{D, \hat{x}, id\}$ GHW representation - (compare with 2.2.1 Proposition in [17]). ψ-Heisenberg-Weyl exponential commutation relations read:

$$exp\{t\partial_\psi\} exp\{a\hat{x}_\psi\} = exp\{at\} exp\{a\hat{x}_\psi\} exp\{t\partial_\psi\}. \tag{2.2.5}$$

To this end let us introduce a pertinent ψ-multiplication $*_\psi$ of functions as specified below.

Notation 2.2.
$\mathrm{x} *_\psi \mathrm{x}^n = \hat{x}_\psi(\mathrm{x}^n) = \frac{(n+1)}{(n+1)_\psi} x^{n+1}; \quad n \geq 0$ hence $\mathrm{x} *_\psi 1 = 1_\psi\ \mathrm{x} \not\equiv \mathrm{x}$
$\mathrm{x}^n *_\psi \mathrm{x} = \hat{x}_\psi^n(\ \mathrm{x}\ ) = \frac{(n+1)!}{(n+1)_\psi!} x^{n+1}; \quad n \geq 0$ hence $1 *_\psi \mathrm{x} = 1_\psi\ \mathrm{x} \not\equiv \mathrm{x}$ therefore
$\mathrm{x} *_\psi \alpha 1 = \alpha 1 *_\psi \mathrm{x} = \mathrm{x} *_\psi \alpha = \alpha *_\psi \mathrm{x} = \alpha 1_\psi\ \mathrm{x}$ and $\forall$ x,$\alpha \in$; $f(x) *_\psi \mathrm{x}^n = f(\hat{x}_\psi)\mathrm{x}^n$.

For $k \neq n \quad \mathrm{x}^n *_\psi \mathrm{x}^k \neq \mathrm{x}^k *_\psi \mathrm{x}^n$ as well as $\mathrm{x}^n *_\psi \mathrm{x}^k \neq \mathrm{x}^{n+k}$ - in general i.e. for arbitrary admissible ψ; compare this with $(\mathrm{x} +_\psi a)^n \neq (\ \mathrm{x} +_\psi a)^{n-1}(\mathrm{x} +_\psi a)$.
In order to facilitate in the future formulation of observations accounted for on the basis of ψ-calculus representation of GHW algebra we shall use what follows.

Definition 2.10. *With Notation 2.2 adopted let us define the* $*_\psi$ *powers of* x *according to*
$x^{n*_\psi} \equiv x *_\psi x^{(n-1)*_\psi} = \hat{x}_\psi(x^{(n-1)*_\psi}) = x *_\psi x *_\psi \ldots *_\psi x = \frac{n!}{n_\psi!} x^n; \quad n \geq 0.$

Note that $x^{n*_\psi} *_\psi x^{k*_\psi} = \frac{n!}{n_\psi!} x^{(n+k)*_\psi} \neq x^{k*_\psi} *_\psi x^{n*_\psi} = \frac{k!}{k_\psi!} x^{(n+k)*_\psi}$ for $k \neq n$ and $x^{0*_\psi} = 1$. This noncommutative ψ-product $*_\psi$ is deviced so as to ensure the following observations:

Observation 2.5.

1. $\partial_\psi x^{n*_\psi} = n x^{(n-1)*_\psi}$; $n \geq 0$
2. $\mathbf{exp}_\psi[\alpha x] \equiv \mathbf{exp}\ \{\alpha \hat{x}_\psi\}\mathbf{1}$
3. $\mathbf{exp}\ [\alpha x] *_\psi \mathbf{exp}_\psi\{\beta \hat{x}_\psi\} = \mathbf{exp}_\psi\{[\alpha + \beta]\hat{x}_\psi\}$
4. $\partial_\psi(x^k *_\psi \quad x^{n*_\psi}) = (Dx^k) *_\psi x^{n*_\psi} + x^k *_\psi (\partial_\psi x^{n*_\psi})$ hence
5. $\partial_\psi(f *_\psi g) = (Df) *_\psi g + f *_\psi (\partial_\psi g)$; f, g - formal series

6. $f(\hat{x}_\psi)g(\hat{x}_\psi)\ \mathbf{1} = f(x) *_\psi \tilde{g}(x)\ ;\ \tilde{g}(x) = g(\hat{x}_\psi)\mathbf{1}$.

Now the consequences of Leibnitz rule (e) for difference-ization of the product are easily feasible. For example the ψ-Poisson process distribution $\mathrm{p}_m(x)$;
$\sum_{m\geq 0} \mathrm{p}_m(x) = 1$;

$$\mathrm{p}_m(x) = \frac{(\lambda x)^m}{m!} *_\psi \mathbf{exp}_\psi[-\lambda x] \tag{2.2.6}$$

is the unique solution of its corresponding ∂_ψ-difference equation

$$\partial_\psi \mathrm{p}_m(x) + \lambda \mathrm{p}_m(x) = \lambda \mathrm{p}_{m-1}(x) m > 0\ ;\ \partial_\psi \mathrm{p}_0(x) = -\lambda \mathrm{p}_0(x) \tag{2.2.7}$$

As announced - the rules of ψ -product $*_\psi$ are accounted for - as a matter of fact - on the basis of ψ-calculus representation of GHW algebra. Indeed; it is enough to consult Observation 2.5 and to introduce ψ-Pincherle derivation $\hat{\partial}_\psi$ of series in powers of the symbol $\hat{x}_\psi$ as below. Then the correspondence between generic relative formulas turns out evident.

Observation 2.6. Let $\hat{\partial}_\psi \equiv \frac{\partial}{\partial \hat{x}_\psi}$ be defined according to $\hat{\partial}_\psi f(\hat{x}_\psi) = [\partial_\psi, f(\hat{x}_\psi)]$. Then $\hat{\partial}_\psi \hat{x}_\psi^n = n\hat{x}_\psi^{n-1}\ ;\ n \geq 0$ and $\hat{\partial}_\psi \hat{x}_\psi^n\ \mathbf{1} = \partial_\psi x^{n*_\psi}$ hence $[\hat{\partial}_\psi f(\hat{x}_\psi)]\mathbf{1}\ = \partial_\psi f(x)$ where f is a formal series in powers of $\hat{x}_\psi$ or equivalently in $*_\psi$ powers of x.

As an example of application note how the solution of 2.2.7 is obtained from the obvious solution $_m(\hat{x}_\psi)$ of the $\hat{\partial}_\psi$-Pincherle differential equation 2.2.8 formulated within G-H-W algebra generated by $\{\partial_\psi, \hat{x}_\psi, id\}$

$$\hat{\partial}_{\psi\, m}(\hat{x}_\psi) + \lambda_m(\hat{x}_\psi) = \lambda_{m-1}(\hat{x}_\psi)\ m > 0\ ;\ \partial_{\psi\, 0}(\hat{x}_\psi) = -\lambda_0(\hat{x}_\psi) \tag{2.2.8}$$

Namely : due to Observation 2.5 (f) $\mathrm{p}_m(x) = {}_m(\hat{x}_\psi)\mathbf{1}$, where

$${}_m(\hat{x}_\psi) = \frac{(\lambda \hat{x}_\psi)^m}{m!} \mathbf{exp}_\psi[-\lambda \hat{x}_\psi]. \tag{2.2.9}$$

3 The general picture

The general picture from the title above relates to the general picture of the algebra $End(P)$ of operators on P as in the following we shall consider the algebra P of polynomials $P = \mathbf{F}[\mathrm{x}]$ over the field $\mathbf{F}$ of characteristic zero.

We shall draw an over view picture of the situation distinguished by possibility to develop umbral calculus for *any* polynomial sequences $\{p_n\}_o^\infty$ instead of those of traditional binomial type only.

In 1901 it was proved [19] that every linear operator mapping P into P may be represented as infinite series in operators $\hat{x}$ and D. In 1986 the authors of [20] supplied the explicit expression for such series in most general case of polynomials in one variable (for many variables see: [21]). Thus according to Proposition 1 from [20] one has:

Proposition 3.1. Let Q be a linear operator that reduces by one each polynomial. Let $\{q_n(\hat{x})\}_{n\geq 0}$ be an arbitrary sequence of polynomials in the operator $\hat{x}$. Then $\hat{T} = \sum_{n\geq 0} q_n(\hat{x})Q^n$ defines a linear operator that maps polynomials into polynomials. Conversely, if $\hat{T}$ is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form

$$\hat{T} = \sum_{n\geq 0} q_n(\hat{x})Q^n.$$

It is also a rather matter of an easy exercise to prove the Proposition 2 from [20]:

Proposition 3.2. *"Let Q be a linear operator that reduces by one each polynomial. Let $\{q_n(\hat{x})\}_{n\geq 0}$ be an arbitrary sequence of polynomials in the operator $\hat{x}$. Let a linear operator that maps polynomials into polynomials be given by*

$\hat{T} = \sum\limits_{n\geq 0} q_n(\hat{x}) Q^n.$

Let $P(x;\lambda) = \sum\limits_{n\geq 0} q_n(x)\lambda^n$ denotes indicator of $\hat{T}$. Then there exists a unique formal series $\Phi(x;\lambda)$; $\Phi(0;\lambda) = 1$such that:

$Q\Phi(x;\lambda) = \lambda\Phi(x;\lambda)$.

Then also $P(x;\lambda) = \Phi(x;\lambda)^{-1}\hat{T}\Phi(x;\lambda)$.

Example 3.1. Note that $\partial_\psi \mathbf{exp}_\psi\{\lambda x\} = \lambda \mathbf{exp}_\psi\{\lambda x\}$; $\mathbf{exp}_\psi[x]|_{x=0} = 1$. (*)
Hence for indicator of $\hat{T}$; $\hat{T} = \sum\limits_{n\geq 0} q_n(\hat{x})\partial_\psi^n$ we have:

$P(x;\lambda) = [\boldsymbol{exp}_\psi\{\lambda x\}]^{-1}\hat{T}\boldsymbol{exp}_\psi\{\lambda x\}$. (**)

After choosing $\psi_n(q) = [n_q!]^{-1}$ we get $\boldsymbol{exp}_\psi\{x\} = \boldsymbol{exp}_q\{x\}$. In this connection note that $exp_o(x) = \frac{1}{1-x}$ and $exp(x)$ are mutual limit deformations for $|x| < 1$ due to:
$\frac{exp_o(z)-1}{z} = exp_o(z) \Rightarrow exp_o(z) = \frac{1}{1-z} = \sum\limits_{k=0}^{\infty} z^k$; $|z| < 1$ i.e.

$$exp(x) \underset{1\leftarrow q}{\longleftarrow} exp_q(x) = \sum_{n=0}^{\infty}\frac{x^n}{n_q!} \underset{q\to 0}{\longrightarrow} \frac{1}{1-x}.$$

Therefore corresponding specifications of (*) such as $exp_o(\lambda x) = \frac{1}{1-\lambda x}$ or $exp(\lambda\mathrm{x})$ lead to corresponding specifications of (**) for divided difference operator ∂_0 and D operator including special cases from [20].

To be complete let us still introduce [2, 3] an important operator $\hat{x}_{Q(\partial_\psi)}$ dual to $Q(\partial_\psi)$.

Definition 3.1. *(see Definition 2.5)*
Let $\{p_n\}_{n\geq 0}$ be the ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator $Q(\partial_\psi)$. A linear map $\hat{x}_{Q(\partial_\psi)}$: $P \to P$; $\hat{x}_{Q(\partial_\psi)} = \frac{(n+1)}{(n+1)_\psi}p_{n+1}$; $n \geq 0$ is called the operator dual to $Q(\partial_\psi)$.

Comment 3.1. Dual in the above sense corresponds to adjoint in ψ-umbral calculus language of linear functionals' umbral algebra (compare with Proposition 1.1.21 in [22]).

It is now obvious that the following holds.

Proposition 3.3. Let $\{q_n(\hat{x}_{Q(\partial_\psi)})\}_{n\geq 0}$ be an arbitrary sequence of polynomials in the operator $\hat{x}_{Q(\partial_\psi)}$. Then $T = \sum\limits_{n\geq 0} q_n(\hat{x}_{Q(\partial_\psi)})Q(\partial_\psi)^n$ defines a linear operator that maps polynomials into polynomials. Conversely, if T is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form

$$T = \sum_{n\geq 0} q_n(\hat{x}_{Q(\partial_\psi)})Q(\partial_\psi)^n. \tag{3.3.1}$$

Comment 3.2. The pair $Q(\partial_\psi)$, $\hat{x}_{Q(\partial_\psi)}$ of dual operators is expected to play a role in the description of quantum-like processes apart from the q-case now vastly exploited [2, 3].

Naturally the Proposition 3.2 for $Q(\partial_\psi)$ and $\hat{x}_{Q(\partial_\psi)}$ dual operators is also valid.
Summing up: we have the following picture for $End(P)$ - the algebra of all linear operators acting on the algebra P of polynomials.

$$Q(P) \equiv \bigcup_{Q} \sum\nolimits_{Q} \subset End(P)$$

and of course $Q(P) \neq End(P)$ where the subfamily $Q(P)$ (with zero map) breaks up into sum of subalgebras $\sum_{Q}$ according to commutativity of these generalized difference-tial operators Q (see Definition 2.4 and Observation 2.2). Also to each subalgebra $\sum_{\psi}$ i.e. to each $Q\left(\partial_{\psi}\right)$ operator there corresponds its dual operator $\hat{x}_{Q\left(\partial_{\psi}\right)}$

$$\hat{x}_{Q\left(\partial_{\psi}\right)} \notin \sum\nolimits_{\psi}$$

and both $Q\left(\partial_{\psi}\right)$ & $\hat{x}_{Q\left(\partial_{\psi}\right)}$ operators are sufficient to build up the whole algebra $End(P)$ according to unique representation given by (3.3.1) including the ∂_{ψ} and $\hat{x}_{\psi}$ case. Summarising: for any admissible ψ we have the following general statement.

General statement:

$$End(P) = [\{\partial_{\psi}, \hat{x}_{\psi}\}] = [\{Q\left(\partial_{\psi}\right)\ ,\ \hat{x}_{Q\left(\partial_{\psi}\right)}\}] = [\{Q\ ,\ \hat{x}_{Q}\}]$$

i.e. the algebra $End(P)$ is generated by any dual pair $\{Q\ ,\ \hat{x}_{Q}\}$ including any dual pair $\{Q\left(\partial_{\psi}\right)$, $\hat{x}_{Q\left(\partial_{\psi}\right)}\}$ or specifically by $\{\partial_{\psi}, \hat{x}_{\psi}\}$ which in turn is determined by a choice of any admissible sequence ψ.

As a matter of fact and in another words: we have bijective correspondences between different commutation classes of ∂_{ψ}-shift invariant operators from $End(P)$, different abelian subalgebras $\sum_{\psi}$, distinct ψ-representations of GHW algebra, different ψ-representations of the reduced incidence algebra R(L(S)) - isomorphic to the algebra Φ_{ψ} of ψ-exponential formal power series [2] and finally - distinct ψ-umbral calculi [7, 11, 14, 23, 2]. These bijective correspondences may be naturally extended to encompass also Q-umbral calculi, Q-representations of GHW algebra and abelian subalgebras $\sum_{Q}$.

(Recall: R(L(S)) is the reduced incidence algebra of L(S) where
L(S)={A; A⊂S; |A| $< \infty$}; S is countable and (L(S); ⊆) is partially ordered set ordered by inclusion [10, 2]).

This is the way the Rota's devise has been carried into effect. The devise *"much is the iteration of the few"* [10] - much of the properties of literally *all* polynomial sequences - as well as GHW algebra representations - is the application of few basic principles of the ψ-umbral difference operator calculus [2].

$\psi-$ **Integration Remark :**

Recall : $\partial_{o}x^{n} = x^{n-1}$. ∂_{o} is identical with divided difference operator. ∂_{o} is identical with ∂_{ψ} for $\psi = \{\psi\left(q\right)_{n}\}_{n\geq 0}$; $\psi\left(q\right)_{n} = 1$; $n \geq 0$. Let $\hat{Q}f(x)f(qx)$.

Recall also that there corresponds to the "∂_{q} difference-ization" the q-integration [24, 25, 26] which is a right inverse operation to "q-difference-ization". Namely

$$F\left(z\right) :\equiv \left(\int_{q} \varphi\right)\left(z\right) := \left(1-q\right) z \sum_{k=0}^{\infty} \varphi\left(q^{k} z\right) q^{k} \tag{3.3.2}$$

i.e.

$$F\left(z\right) \equiv \left(\int_{q} \varphi\right)\left(z\right) = \left(1-q\right) z \left(\sum_{k=0}^{\infty} q^{k} \hat{Q}^{k} \varphi\right)\left(z\right) = \left(\left(1-q\right) z \frac{1}{1-q\hat{Q}} \varphi\right)\left(z\right). \tag{3.3.3}$$

Of course

$$\partial_{q} \circ \int_{q} = id \tag{3.3.4}$$

as

$$\frac{1-q\hat{Q}}{\left(1-q\right)} \partial_{0} \left(\left(1-q\right) \hat{z} \frac{1}{1-q\hat{Q}}\right) = id. \tag{3.3.5}$$

Naturally (3.3.5) might serve to define a right inverse operation to "q-difference-ization"

$$(\partial_q \varphi)(x) = \frac{1-q\hat{Q}}{(1-q)} \partial_0 \varphi(x)$$

and consequently the "q-integration " as represented by (3.3.2) and (3.3.3). As it is well known the definite q-integral is an numerical approximation of the definite integral obtained in the $q \to 1$ limit. Following the q-case example we introduce now an R-integration (consult Remark 2.1).

$$\int_R x^n = \left(\hat{x} \frac{1}{R\left(q\hat{Q}\right)} \right) x^n = \frac{1}{R\left(q^{n+1}\right)} x^{n+1}; \quad n \geq 0 \tag{3.3.6}$$

Of course $\partial_R \circ \int_R = id$ as

$$R\left(q\hat{Q}\right) \partial_o \left(\hat{x} \frac{1}{R\left(q\hat{Q}\right)} \right) = id. \tag{3.3.7}$$

Let us then finally introduce the analogous representation for ∂_ψ difference-ization

$$\partial_\psi = \hat{n}_\psi \partial_o; \ \hat{n}_\psi x^{n-1} = n_\psi x^{n-1}; \ n \geq 1. \tag{3.3.8}$$

Then

$$\int_\psi x^n = \left(\hat{x} \frac{1}{\hat{n}_\psi} \right) x^n = \frac{1}{(n+1)_\psi} x^{n+1}; \ n \geq 0 \tag{3.3.9}$$

and of course

$$\partial_\psi \circ \int_\psi = id \tag{3.3.10}$$

Closing Remark:

The picture that emerges discloses the fact that any ψ-representation of finite operator calculus or equivalently - any ψ-representation of GHW algebra makes up an example of the algebraization of the analysis - naturally when constrained to the algebra of polynomials. We did restricted all our considerations to the algebra P of polynomials. Therefore the distinction in-between difference and differentiation operators disappears. All linear operators on P are both difference and differentiation operators if the degree of differentiation or difference operator is unlimited. For example $\frac{d}{dx} = \sum_{k \geq 1} \frac{d_k}{k!} \Delta^k$ where $d_k = \left[\frac{d}{dx} x^{\underline{k}}\right]_{x=0} = (-1)^{k-1}(k-1)!$ or $\Delta = \sum_{n \geq 1} \frac{\delta_n}{n!} \frac{d^n}{dx^n}$ where $\delta_n = [\Delta x^n]_{x=0} = 1$. Thus the difference and differential operators and equations are treated on the same footing.

An interesting task (which seems to be still ahead) is to investigate the Q representation of finite operator calculus as an example of the algebraization of the analysis - naturally when constrained to the algebra of polynomials.

4 Glossary

Here now come short indicatory glossaries of terms and notation used by Ward [1], Viskov [6, 7], Markowsky [11], Roman [27]- [31] on one side and the Rota-oriented notation on the other side.

Ward	**Rota - oriented** (this note)
$[n]; \ [n]!$	$n_\psi; \ n_\psi!$
basic binomial coefficient $[n,r] = \frac{[n]!}{[r]![n-r]!}$	ψ-binomial coefficient $\binom{n}{k}_\psi \equiv \frac{n_\psi^{\underline{k}}}{k_\psi!}$

Ward	**Rota - oriented** (this note)
$D = D_x$ - the operator D $D\,x^n = [n]\;x^{n-1}$	∂_ψ - the ψ-derivative $\partial_\psi\,x^n = n_\psi\,x^{n-1}$
$(x+y)^n$ $(x+y)^n \equiv \sum_{r=0}^{n}[n,r]\,x^{n-r}y^r$	$(x+_\psi y)^n$ $(x+_\psi y)^n = \sum_{k=0}^{n}\binom{n}{k}_\psi x^k y^{n-k}$
basic displacement symbol $E^t;\ t \in \mathbf{Z}$ $E\varphi(x) = \varphi(x+1)$ $E^t\varphi(x) = \varphi\left(x+\bar{t}\right)$	generalized shift operator $E^y\,(\partial_\psi) \equiv exp_\psi\{y\partial_\psi\};\ y \in \mathbf{F}$ $E(\partial_\psi)\varphi(x) = \varphi(x+_\psi 1)$ $E^y(\partial_\psi)x^n \equiv (x+_\psi y)^n$
basic difference operator $\Delta = E - id$ $\Delta = \varepsilon(D) - id = \sum_{n=0}^{\infty}\frac{D^n}{[n]!} - id$	ψ-difference delta operator $\Delta_\psi = E^y(\partial_\psi) - id$

Roman	**Rota - oriented** (this note)
$t;\ tx^n = nx^{n-1}$ $\langle t^k \vert p(x)\rangle = p^{(k)}(0)$	∂_ψ - the ψ-derivative $\partial_\psi x^n = n_\psi x^{n-1}$ $[\partial_\psi^k p(x)]\vert_{x=0}$
evaluation functional $\epsilon_y(t) = \exp\{yt\}$ $\langle t^k \vert x^n\rangle = n!\delta_{n,k}$ $\langle \epsilon_y(t) \vert p(x)\rangle = p(y)$ $\epsilon_y(t)x^n = \sum_{k\geq 0}\binom{n}{k}x^k y^{n-k}$	generalized shift operator $E^y(\partial_\psi) = \exp_\psi\{y\partial_\psi\}$ $[E^y(\partial_\psi)p_n(x)]\vert_{x=0} = p_n(y)$ $E^y(\partial_\psi)p_n(x) = \sum_{k\geq 0}\binom{n}{k}_\psi p_k(x)p_{n-k}(y)$

Roman	**Rota - oriented** (this note)
formal derivative $f'(t) \equiv \frac{d}{dt} f(t)$ $\overline{f}(t)$ compositional inverse of formal power series $f(t)$	Pincherle derivative $[Q(\partial_\psi)]' \equiv \frac{d}{d\partial_\psi} Q(\partial_\psi)$ $Q^{-1}(\partial_\psi)$ compositional inverse of formal power series $Q(\partial_\psi)$
$\theta_t;\ \ \theta_t x^n = x^{n+1};\ n \geq 0$ $\theta_t t = \hat{x} D$	$\hat{x}_\psi;\ \ \hat{x}_\psi x^n = \frac{n+1}{(n+1)_\psi} x^{n+1};\ n \geq 0$ $\hat{x}_\psi \partial_\psi = \hat{x} D = \hat{N}$
$\sum\limits_{k \geq 0} \frac{s_k(x)}{k_\psi!} t^k =$ $[g(\overline{f}(z))]^{-1} \exp\{x\overline{f}(t)\}$ $\{s_n(x)\}_{n \geq 0}$ - Sheffer sequence for $(g(t), f(t))$	$\sum\limits_{k \geq 0} \frac{s_k(x)}{k_\psi!} z^k =$ $s(q^{-1}(z)) \exp_\psi \{x q^{-1}(z)\}$ $q(t)$, $s(t)$ indicators of $Q(\partial_\psi)$ and S_{∂_ψ}
$g(t)\, s_n(x) = q_n(x)$ - sequence associated for $f(t)$	$s_n(x) = S_{\partial_\psi}^{-1}\, q_n(x)$ - ∂_ψ - basic sequence of $Q(\partial_\psi)$
The expansion theorem: $h(t) = \sum\limits_{k=0}^{\infty} \frac{\langle h(t) \vert p_k(x)\rangle}{k!} f(t)^k$ $p_n(x)$ - sequence associated for $f(t)$	The First Expansion Theorem $T_{\partial_\psi} = \sum\limits_{n \geq 0} \frac{[T_{\partial_\psi} p_n(z)]\vert_{z=0}}{n_\psi} Q(\partial_\psi)^n$ ∂_ψ - basic polynomial sequence $\{p_n\}_0^\infty$
$\exp\{y\overline{f}(t)\} = \sum\limits_{k=0}^{\infty} \frac{p_k(y)}{k!} t^k$	$\exp_\psi\{xQ^{-1}(x)\} = \sum\limits_{k \geq 0} \frac{p_k(y)}{k!} z^k$
The Sheffer Identity: $s_n(x+y) = \sum\limits_{k=0}^{n} \binom{n}{k} p_n(y) s_{n-k}(x)$	The Sheffer ψ-Binomial Theorem: $s_n(x +_\psi y) = \sum\limits_{k \geq 0} \binom{n}{k}_\psi s_k(x) q_{n-k}(y)$

Viskov	**Rota - oriented** (this note)

Viskov	**Rota - oriented** (this note)
θ_ψ - the ψ-derivative $\theta_\psi\, x^n = \frac{\psi_{n-1}}{\psi_n} x^{n-1}$	∂_ψ - the ψ-derivative $\partial_\psi\, x^n = n_\psi\, x^{n-1}$
$A_p\ (p = \{p_n\}_0^\infty)$ $A_p\, p_n = p_{n-1}$	Q $Q\, p_n = n_\psi p_{n-1}$
$B_p\ (p = \{p_n\}_0^\infty)$ $B_p\, p_n = (n+1)\, p_{n+1}$	$\hat{x}_Q$ $\hat{x}_Q\, p_n = \frac{n+1}{(n+1)_\psi} p_{n+1}$
$E_p^y\ (p = \{p_n\}_0^\infty)$ $E_p^y\, p_n(x) = \sum_{k=0}^{n} p_{n-k}(x) p_k(y)$	$E^y\,(\partial_\psi) = exp_\psi\{y\partial_\psi\}$ $E^y\,(\partial_\psi)\ p_n(x) =$ $= \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x) p_{n-k}(y)$
$T - \varepsilon_p$-operator: $T\, A_p = A_p\, T$	E^y - shift operator: $E^y \varphi(x) = \varphi(x +_\psi y)$
$\forall_{y\in F}\ T E_p^y = E_p^y T$	T - ∂_ψ-shift invariant operator: $\forall_{\alpha\in F}\ [T, E^\alpha(\partial_\psi)] = 0$
Q - δ_ψ-operator: Q - ϵ_p-operator and $Qx = const \neq 0$	$Q(\partial_\psi)$ - ∂_ψ-delta-operator: $Q(\partial_\psi)$ - ∂_ψ-shift-invariant and $Q(\partial_\psi)(id) = const \neq 0$
$\{p_n(x), n \geq 0\}$ - (Q, ψ)-basic polynomial sequence of the δ_ψ-operator Q	$\{p_n\}_{n\geq 0}$ -∂_ψ-basic polynomial sequence of the ∂_ψ-delta-operator $Q(\partial_\psi)$

Viskov	**Rota - oriented** (this note)
ψ-binomiality property $\Psi_y s_n(x) =$ $= \sum_{m=0}^{n} \frac{\psi_n \psi_{n-m}}{\psi_n} s_m(x) p_{n-m}(y)$	ψ-binomiality property $E^y(\partial_\psi) p_n(x) =$ $= \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x) p_{n-k}(y)$
$T = \sum_{n\geq 0} \psi_n [VTp_n(x)] Q^n$ $T\Psi_y p(x) =$ $\sum_{n\geq 0} \psi_n s_n(y) Q^n STp(x)$	$T = \sum_{n\geq 0} \frac{[Tp_n(z)]\|_{z=0}}{n_\psi!} Q(\partial_\psi)^n$ $Tp(x +_\psi y) =$ $\sum_{k\geq 0} \frac{s_k(y)}{k_\psi!} Q(\partial_\psi)^k S_{\partial_\psi} Tp(x)$

Markowsky	**Rota - oriented** (this note)
L - the differential operator $L\, p_n = p_{n-1}$	Q $Q\, p_n = n_\psi p_{n-1}$
M $M\, p_n = p_{n+1}$	$\hat{x}_Q$ $\hat{x}_Q\, p_n = \frac{n+1}{(n+1)_\psi} p_{n+1}$
L_y $L_y\, p_n(x) =$ $= \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y)$	$F^y\,(Q) = \sum_{k\geq 0} \frac{p_k(y)}{k_\psi!} Q^k$ $F^y\,(Q)\ p_n(x) =$ $= \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x) p_{n-k}(y)$
E^a - shift-operator: $E^a\, f(x) = f(x+a)$	E^y - ∂_ψ-shift operator: $E^y \varphi(x) = \varphi(x +_\psi y)$
G - shift-invariant operator: $EG = GE$	T - ∂_ψ-shift invariant operator: $\forall_{\alpha\in F}\ [T, E^\alpha(\partial_\psi)] = 0$

Markowsky	**Rota - oriented** (this note)
G - delta-operator: G - shift-invariant and $Gx = const \neq 0$	$Q(\partial_\psi)$ - ∂_ψ-delta-operator: $Q(\partial_\psi)$ - ∂_ψ-shift-invariant and $Q(\partial_\psi)(id) = const \neq 0$
$D_L(G)$ L - Pincherle derivative of G $D_L(G) = [G, M]$	$G' = [G(Q), \hat{x}_Q]$ Q - Pincherle derivative
$\{Q_0, Q_1, ...\}$ - basic family for differential operator L	$\{p_n\}_{n\geq 0}$ -ψ-basic polynomial sequence of the generalized difference operator Q
binomiality property $P_n(x+y) =$ $= \sum_{i=0}^{n} \binom{n}{i} P_i(x) P_{n-1}(y)$	Q - ψ-binomiali property $F^y(Q)p_n(x) =$ $= \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x) p_{n-k}(y)$

Acknowledgements

Discussions with Ewa Grądzka are highly acknowledged. The author expresses his gratitude to her also for preparation of the LaTeX version of this contribution.

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GEOMETRY OF LAGRANGIAN MANIFOLDS IN THERMODYNAMICS

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Abstract

It is considered, that the classical thermodynamic properties of substance are defined by relations connecting volume, pressure, temperature, entropy and energy of the given substance. Generally substance is characterized by some number of magnitudes, which half is intensive, and half — by extensive magnitudes. From this point of view pressure and temperature are considered as intensive magnitudes, and volume and entropy - extensive magnitudes. The modern point of view consists that in a condition of a thermodynamic equilibrium the substance should be characterized by a point in the space $R^{2n+1}(p,q,\Phi)$, where coordinates $q = (q_1, q_2, \dots, q_n)$ are intensive magnitudes, first two of which are pressure and temperature ($q_1 = P, \quad q_2 = -T$), and coordinate $p_1, p_2, \dots, p_n$) are extensive coordinates first two from which are volume and entropy ($p_1 = V, \quad p_2 = S$). First four coordinates (P, V, T, S) describe, so to tell, variables of a mechanical nature for homogeneous substances. generally follows to consider heterogeneous (i.e. multicomponent) systems, and also variables not mechanical nature (for example, electromagnetic properties). In any case, the space $R^{2n+1}(p,q,\Phi)$ is supplied by a contact structure, i.e. differential 1-form $\omega = d\Phi - pdq$, and the set of thermodynamic equilibrium states of substance is represented by a submanifold $L \subset R^{2n+1}(p,q,\Phi)$, such that $\omega = 0$. Hence tha projection $L_0 \subset R^{2n}(p,q)$ is a Lagrangian submanifold in symplectic space $R^{2n}(p,q)$ with the symplectic form $\Omega = dp \wedge dq$. Function Φ is function of action on Lagrangian manifold L_0, $d\Phi = pdq$. For classical thermodynamics it coincides with a thermodynamic potential ($\Phi = E + PV - TS$).

By Gibbs ([1]) the energy E is a function of variables (V, S), as, however, and all remaining thermodynamic magnitudes. It hence, that Lagrangian manifold L_0 bijectively is projected on a domain in the space $R^2(P,S)$, i.e. the manifold L is defined by the graph of function $E = E(V,S)$.

Implicitly Gibbs actually assumed, that the surface $E = E(V,S)$, being noncompact, its any plane of support has by property, that touches a surface in each common point. This condition ensures realization of the following statement: from minimization thermodynamic potential at fixed P and T the positiveness of Hessian of function $E = E(V,S)$, $\mathrm{Hess}_{(V,S)}E(V,S) > 0$ follows. By Maslov ([3]) such condition are called essential. Then in essential condition are fulfilled local thermodynamic inequalities ([2]). Let's consider function

$$\tilde{\Phi}^L(q) = \min_{q(x)=q; x \in L} \Phi(x),$$

under condition of existence of the minimum in question. Consider a symplectic transformation φ of symplectic spaces

$$\varphi : R^{2n}(p,q) \longrightarrow R^{2n}(P,Q)$$

and Lagrangian manifold

$\Gamma_\varphi \subset R_{4n}(P,p,Q,q)$, which is the graph of transformations φ. Let S - be function of action on Lagrangian manifold Γ_φ, $dS = PdQ - pdq$. Let's assume, that manifold Γ_φ is uniquely projected on the space $R^{2n}(Q,q)$. Then function S can be understood as function

of variables (Q, q), $S = S(Q, q)$. In this case the function S is called generating function of transformation φ.

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be defined similar to manifold L_1.

Theorem 1. *At an approaching choice of boundary conditions on manifold L and transformation φ the following formula takes place*

$$\tilde{\Phi}^{L_1}(Q) = \min_q \left(S(Q, q) + \tilde{\Phi}^L(q) \right).$$

Similar, if $\varphi_1, \varphi_2, \ldots, \varphi_n$ is a sequence of symplectic transformations which admit generating functions

$$S_1(Q, q_1), S_2(q_1, q_2), \ldots, S_n(q_{n-1}, q_n),$$

and $L_1 = \tilde{\varphi}_1 \tilde{\varphi}_2 \cdots \tilde{\varphi}_n(L)$, then

$$\tilde{\Phi}^{L_1}(Q) = \min_{q_1, q_2, \ldots, q_n} \left(S_1(Q, q_1) + S_2(q_1, q_2) + \cdots + S_n(q_{n-1}, q_n) + \tilde{\Phi}^L(q_n) \right).$$

The choice of boundary conditions should supply existence of a minimum at an evaluation of $\tilde{\Phi}^{L_1}$. The theorem 1 supplies the map of insignificant points of manifold L into insignificant points of manifold L_1 (compare [3]).

the Lagrangian L.

Theorem 2. *Assume that the Lagrangian $L(q, \dot{q})$ satisfies the conditions that the index of inertia of $Hess_{q,\dot{q}} L$ equals to (n, n). Let $S(Q, q, t)$ be the generating function of transformation induced by Lagrangian L. Assume that there is a minimum*

$$\tilde{\Phi}^{L_1}(Q) = \min_q \left(S(Q, q) + \tilde{\Phi}^L(q) \right).$$

Then

$$Hess_Q \tilde{\Phi}^{L_1}(Q) < 0.$$

The theorem 2 ensures realization local thermodynamic inequalities in essential points of Lagrange manifold L_1.

Theorem 3. *Let L be a Lagrange manifold which uniquely projected onto p–coordinates. Then at an approaching choice of boundary conditions in essential points the local thermodynamic inequalities are fulfilled, i.e.*

$$Hess_q \Phi^L(q) < 0.$$

As approaching boundary conditions for the theorem 3 the following condition can serve:
Condition 1. *The function $E(p)$, $dE = -qdp$, $E = \Phi^L(p) - pq$ is locally convex upwards in all domain of definition $G(p) \subset R^n(p)$ behind elimination of some compact set $K \subset G(p)$.*

The condition 1 is fulfilled for the majority of modelling examples of gases (ideal gas, Van der Waals gas , degenerated Fermi gas).

The theorems 1 and 2 allow to construct such Lagrangian manifolds, which are not projected uniquely on p - coordinate, but in all essential points satisfie to local thermodynamic inequalities..

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GEOMETRY OF GOURSAT FLAGS AND THEIR SINGULARITIES OF CODIMENSION 2

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April 9, 2001

1 Basic geometry – geometric classes – Jean's strata

With every Goursat distribution – a particular rank–2 subbundle D in the tangent bundle to an n–dimensional manifold M (C^∞ or analytic; $n \geq r+2$) such that the Lie square $[D,\, D]$ of D is everywhere of rank 3, the Lie square of $[D,\, D]$ is everywhere of rank 4, and so on until obtaining the full TM – associated is its *flag* of ascending induced subbundles $D^r = D,\ D^{r-1} = [D,\, D],\ D^{r-2} = [[D,\, D],\ [D,\, D]], \dots, D^0 = TM$ indexed by their *coranks* assumed to be constant independently of a point in M. The *length* of such a flag is r.

This is a very restrictive condition and G. germs are (excepting $n = 4$ and $r = 2$ – the classical situation of Engel, 1889) of codimension ∞ among germs of all distributions of fixed rank and corank. Yet, there exists an interesting trade off – the absence of functional parameters in local preliminary normal forms.

In fact, Goursat distributions of corank r locally admit polynomial presentations of degrees $\leq r-1$ of Kumpera and Ruiz [KRu], using only real parameters, in numbers not exceeding $r-3$, many of them possibly redundant. They also admit a trigonometric presentation springing from the kinematic model of a car pulling $r-1$ passive trailers, developed by several authors in the 90s and refined to its limits by Jean [J] bringing in critical angles $a_1 = \frac{\pi}{2},\ a_{j+1} = \arctan(\sin a_j)$). As a matter of fact, Kumpera and Ruiz discovered *singularities* hidden in flags of Goursat distributions. First attempts at defining them in a coordinate-free way were made in [BH] (p. 455), then in [CMPRe]. In [MonZ] singularities of Goursat flags were described in a canonical way, with a consistent use of the associated subflag of Cauchy–characteristic subdistributions.

Recalling, the basic singular features in the car' presentation (attention: in that model, the last trailer has number 0, while trailer hooked to the car – number $r-2$, the car itself has number $r-1$) are possible right angles between neighbouring trailers No $k-1$ and k. They correspond to *coalescences*, at a point, of flag' member D^{k+1} with the Cauchy–characteristic directions of two-step bigger member D^{k-1}. But flags exhibit also higher order singularities, implicitly present already in [J] (and constituting its strength), explicitly called *tangent* in [MonZ].

After [J], [CM] (last chap. 6), [MonZ] it is known that germs of G. flags of length r (or: distributions of corank r) can be stratified into F_{2r-3} (Fibonacci number) *geometric classes* encoded by words of length r over the alphabet $\{\mathrm{G}, \mathrm{S}, \mathrm{T}\}$ s. t. two first letters are always G and never a T goes directly after a G.

A letter S is written in the code when a basic geometric coalescence takes place for the corresponding flag' member D^k, members being indexed, we recall, backwards (or else: the right angle a_1 occurs between two neighbouring trailers). A sequence STT. . . T inside a code corresponds to the next (i. e., closer to the car) neighbouring trailer making angle a_2 with the second in the couple having the right angle, still next making angle a_3 with the first next, and so as long as many T's is in the sequence. This singular behaviour geometrically means that D^{k+1} is tangent at the reference point to the *locus* of the previous singularity 'D^k in basic singular position', plus D^{k+2} is tangent at that point to the locus of the singularity 'ST', plus D^{k+3} is tangent at that point to 'STT', and so on.

At the one next step the angle rule, or consecutive tangencies rule, breaks down and now the letters G are written in row until some next S (related to a new right angle in the configuration of trailers) appears.

Jean's strata (the materializations of geometric classes on a given manifold carrying a G. flag) are, when non-empty, regular embedded submanifolds of codimensions that are easily computable. Namely, the codimension of a stratum having code $\mathcal{C}$ is equal to the number of letters S and T in $\mathcal{C}$ (cf. [M5], Sec. 1.4). The only codimension–0 stratum GGG. . . G, being open and dense, is non-empty on any n–dimensional manifold carrying a flag; G. germs at its points are equivalent to the classical *chained model* of von Weber (1898) – Cartan (1914) – Goursat (1922) featuring no extra parameters:

$$\begin{aligned} dx^2 - x^3 dx^1 &= 0\,, \\ dx^3 - x^4 dx^1 &= 0\,, \\ dx^4 - x^5 dx^1 &= 0\,, \\ * \qquad & \quad * \\ dx^{r+1} - x^{r+2} dx^{k+1} &= 0 \end{aligned}$$

(it should be understood as the germ at $0 \in \mathbb{R}^n(x^1, x^2, \ldots, x^n)$).

The germs at points of the codimension–1 strata GG. . . GSG. . . G have been classified (for arbitrary length) in [M2], although only in [M3] the *geometric* description – based on [MonZ] – of the conditions securing the singular models was given. These singularities are *simple* in the singularity theory sense; parameters of K-R can be eliminated. The only invariant is the position $3 \le k \le r$ of the unique letter S in the r-letter code (the place in the flag where the unique coalescence of linear spaces at a point occurs). As a representative of the relevant orbit on n–dimensional manifolds can be taken the germ at $0 \in \mathbb{R}^n(x^1, x^2, \ldots, x^n)$ of

$$\begin{aligned} dx^2 - x^3 dx^1 &= 0\,, \\ dx^3 - x^4 dx^1 &= 0\,, \\ * \qquad & \quad * \\ dx^k - x^{k+1} dx^1 &= 0\,, \\ dx^1 - x^{k+2} dx^{k+1} &= 0\,, \\ dx^{k+2} - (1 + x^{k+3}) dx^{k+1} &= 0\,, \\ dx^{k+3} - x^{k+4} dx^{k+1} &= 0\,, \\ * \qquad & \quad * \\ dx^{r+1} - x^{r+2} dx^{k+1} &= 0\,. \end{aligned}$$

In the present report we want to itemize what is known concerning the classification question for singularities of flags of codimension 2. That is – singularities at points in Jean's strata having exactly *two* non–G letters in the code.

2 Strata having ST in their codes.

These strata have been recently classified in [M5] modulo one more singular (of codimension 3) feature of flags. In fact, for the whole strata GGSTG...G, G...GSTG, G...GST and for each GG...GSTG...G (at least three G's in the beginning, at least two G's in the end) *less* certain embedded submanifold of codimension 3 (not definable in the G, S, T language), we derive unique local models, the same in either of the categories C^∞ or analytic. Again, only the position of the sequence ST in the code counts, and these singularities appear to be simple for Goursat flags of arbitrary length. When the letter S is at the k-th place in the code, $4 \le k \le r-3$, the only local model, on a manifold of dimension n, is the germ at $0 \in \mathbb{R}^n(x^1, x^2, \dots, x^n)$ of the rank–2 distribution described by the following r Pfaffian equations

$$\begin{aligned}
dx^2 - x^3 dx^1 &= 0\,,\\
dx^3 - x^4 dx^1 &= 0\,,\\
* \qquad & \quad *\\
dx^k - x^{k+1} dx^1 &= 0\,,\\
dx^1 - x^{k+2} dx^{k+1} &= 0\,,\\
dx^{k+2} - x^{k+3} dx^{k+1} &= 0\,,\\
dx^{k+3} - (1 + x^{k+4}) dx^{k+1} &= 0\,,\\
dx^{k+4} - (1 + x^{k+5}) dx^{k+1} &= 0\,,\\
dx^{k+5} - x^{k+6} dx^{k+1} &= 0\,,\\
* \qquad & \quad *\\
dx^{r+1} - x^{r+2} dx^{k+1} &= 0\,.
\end{aligned}$$

The unique local model for the entire stratum GGSTGG...G is as the above for $k = 3$ except that its $(k+3)$-th, i.e., sixth equation reads $dx^7 - x^8 dx^4 = 0$ instead of $dx^7 - (1 + x^8) dx^4 = 0$. The models for G...GSTG and G...GST are the above for $k = r-2$ and $k = r-1$, respectively, with simplifications due to the fact that only the coordinates up to x^{r+2} enter the local description.

This series of smooth, or analytic, local models consists of certain mentioned in the beginning (but highly specified in the course of a long proof) polynomial presentations of [KRu], of the G. germs in the respective geometric classes. After passing to the (dual) vector fields' writing, polynomials are only of degree 2, because only one flag's member is in a basic singular position, but there is plenty of constants in a preliminary presentation which should either be normalized (to 1 in the occurrence) or annihilated. Among the latter, certain are (much) more resistant. Surprisingly, the reason for that boils down to the arithmetical fact that – only for $k \ge 4$ – there exist natural i's such that $3k - 2 + i$ does not sit in the *additive semigroup* generated by 3 and $3k - 5$. As one can easily check, these values of i are precisely $1, 4, \dots, 1 + 3(k-4)$ (for inst., $i = 1$ and 4 for $k = 5$). They are just instances of the *interesting distances* put forward in [M2] (Def. 4.2 there for $j = 1$; note that that-time-k is now $k - 3$ [1]).

3 Strata having SS in their codes.

The work on these singularities is in progress. There are rather strong indications that the whole strata GGSSG...G are just single orbits of the local classification, with possible

[1] in Thm. 4.1 in [M2] the condition '$(i - 1 \ge k(j+2)$ and $i - 1 \in \mathcal{Z}_{jk})$' should be replaced by '$(i - 1 - k(j+2) \in \mathcal{Z}_{jk})$'

representatives

$$
\begin{aligned}
dx^2 - x^3 dx^1 &= 0\,,\\
dx^3 - x^4 dx^1 &= 0\,,\\
dx^1 - x^5 dx^4 &= 0\,,\\
dx^4 - x^6 dx^5 &= 0\,,\\
dx^6 - (1 + x^7) dx^5 &= 0\,,\\
dx^7 - x^8 dx^5 &= 0\,,\\
dx^8 - x^9 dx^5 &= 0\,,\\
* \qquad & \quad *\\
dx^{r+1} - x^{r+2} dx^5 &= 0\,.
\end{aligned}
$$

Concerning the strata GG…GSSGG…G with at least three G's in the beginning and at least three G's in the end, when the letters S are at the k-th and $(k+1)$-th places in the code, $4 \le k \le r-4$, we **conjecture** that, on an n–dimensional manifold, excepting certain embedded submanifold of codimension 3 (again, not definable in the G, S, T language) sitting in the stratum and cutting it into two disjoint parts, the germs at points of either part are equivalent to precisely one of the couple of germs at $0 \in \mathbb{R}^n(x^1,\, x^2, \dots,\, x^n)$ of

$$
\begin{aligned}
dx^2 - x^3 dx^1 &= 0\,,\\
dx^3 - x^4 dx^1 &= 0\,,\\
* \qquad & \quad *\\
dx^k - x^{k+1} dx^1 &= 0\,,\\
dx^1 - x^{k+2} dx^{k+1} &= 0\,,\\
dx^{k+1} - x^{k+3} dx^{k+2} &= 0\,,\\
dx^{k+3} - (1 + x^{k+4}) dx^{k+2} &= 0\,,\\
dx^{k+4} - x^{k+5} dx^{k+2} &= 0\,,\\
dx^{k+5} - (\pm 1 + x^{k+6}) dx^{k+2} &= 0\,,\\
dx^{k+6} - x^{k+7} dx^{k+2} &= 0\,,\\
* \qquad & \quad *\\
dx^{r+1} - x^{r+2} dx^{k+2} &= 0\,.
\end{aligned}
$$

This should hold in both categories C^∞ and analytic. As of now, this is, we repeat, a conjecture with work on it being in progress.

4 Strata having two not neighbouring S in their codes.

One should *not*, however, suppose that all codimension–2 singularities of Goursat flags are simple (admit only discrete local models). It is already not so in the geometric classes having the sequence SGS in the code, and at least three G's in the beginning, as shown in [M4].

Before reviewing it in more detail, we want to note that first examples of continuous moduli in the Goursat world were found in geometric classes of codimension 3: GGGSTTGGG — [PRe], length 9, and, slightly later, GGSGSGSG — [M1], length 8.[2] The latter example extends naturally and easily – see [M1], Rem. 4 – to the series of geometric classes GGSGSG…SG (when r is even) and GGGSGSG…SG (when r is odd)

[2] Those findings were also briefly reported in [CMPRe].

having precisely modality $m = \left[\frac{r}{2}\right] - 3$ wrt the classification of germs by diffeos acting in the base manifold: the orbits in these classes are exactly parametrized by m different real parameters.
A far-reaching [but, it should be admitted, not yet sufficiently verified on various examples] **conjecture** of 1997 says that $\left[\frac{r}{2}\right] - 3$ is the maximal modality of germs of Goursat flags of length r.

Since the previous (2nd) Krynica Conference, an extensive work has been done on the geometric classes having the sequence SGS in their codes, and qualitatively new features, wrt the basic geometries ST and SS, revealed. In fact, excepting the class GGSGSGGG, a module of local C^∞ or analytic classification was found not later than 3 steps after the behaviour SGS in the flag, see Thm. 1 in [M4]. Any germ in the class GG...GSGSGGG, with the first S being at the place $k \geq 4$, appears equivalent to precisely one of the germs at $0 \in \mathbb{R}^n(x^1, \ldots, x^{k+7}; x^{k+8}, \ldots, x^n)$ of

$$\begin{aligned}
dx^2 - x^3 dx^1 &= 0\,,\\
dx^3 - x^4 dx^1 &= 0\,,\\
* \qquad & *\\
dx^k - x^{k+1} dx^1 &= 0\,,\\
dx^1 - x^{k+2} dx^{k+1} &= 0\,,\\
dx^{k+2} - \left(1 + x^{k+3}\right) dx^{k+1} &= 0\,,\\
dx^{k+1} - x^{k+4} dx^{k+3} &= 0\,,\\
dx^{k+4} - \left(1 + x^{k+5}\right) dx^{k+3} &= 0\,,\\
dx^{k+5} - x^{k+6} dx^{k+3} &= 0\,,\\
dx^{k+6} - \left(c + x^{k+7}\right) dx^{k+3} &= 0
\end{aligned}$$

parametrized by $c \in \mathbb{R}$. This invariant parameter can be better exemplified by *not* reducing to 0 the constant in the one before last Pfaffian equation above. Quoting from Rem. 1 in [M4], in the family of KR pseudo-normal forms (for germs in the geometric class under consideration)

$$\begin{aligned}
dx^2 - x^3 dx^1 &= 0\,,\\
dx^3 - x^4 dx^1 &= 0\,,\\
* \qquad & *\\
dx^k - x^{k+1} dx^1 &= 0\,,\\
dx^1 - x^{k+2} dx^{k+1} &= 0\,,\\
dx^{k+2} - \left(1 + x^{k+3}\right) dx^{k+1} &= 0\,,\\
dx^{k+1} - x^{k+4} dx^{k+3} &= 0\,,\\
dx^{k+4} - \left(1 + x^{k+5}\right) dx^{k+3} &= 0\,,\\
dx^{k+5} - (b + x^{k+6}) dx^{k+3} &= 0\,,\\
dx^{k+6} - \left(c + x^{k+7}\right) dx^{k+3} &= 0\,,
\end{aligned}$$

the quantity $c - 7b - \frac{5}{3}b^2$ is an invariant of the local smooth or analytic conjugacies preserving $0 \in \mathbb{R}^n$.
The analysis of the classes GG...GSGSG...G, but now with at least 4 letters G in the beginning and more letters G in the end, was being continued after closing [M4].

If the first S is at the place $k \geq 5$, then (if the ambient dimension n is big enough) in the next group of 4 letters G there hides itself a new invariant independent of the one discussed above. Thus the modality of the SGS singularities is in many cases at least 2. One can say that the first module is concealed, provided $k \geq 4$, in the group of three generic positions in the flag directly after the SGS behaviour, while the second module – provided $k \geq 5$ – in the group of the following four generic positions. It is plausible that, when k is at least 6 and n is big enough, in certain next group of G's a third module is located. And more, it is not even excluded that in *geometric classes of unbounded length*[3] [GG. . . GSGS], with l letters G in the beginning, modality can be at least $l - 2$.

The geometric classes with sequences SG. . . GS in their codes have been little investigated yet. Nevertheless, the arguments pinpointing moduli in generic prolongations of the geometry SGS seem to guarantee the abundance of numeric invariants among germs featuring the geometries SG. . . GS, too.

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KV-COHOMOLOGY OF CONTACT MANIFOLDS

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Abstract

Given a contact manifold (M, a) the group of a-preserving diffeomorphisms is denoted by $G(a)$.We construct a Koszul-Vinberg chain complex $C(a)$ on which $G(a)$ acts by chain-complex homomorphism. The $G(a)$-equivariant cohmology spaces of $C(a)$ produce new contact invariants.

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ON THE LEAVES OF A PREFOLIATION OF A K–DIFFERENTIAL SPACE

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March 7, 2001

Two years ago at the conference in Krynica, prof. W. Waliszewski has presented the definition of a $\mathbb{K}$-differential space which is a generalization of the notions of a differentiable manifold and a differential space in the sense of Sikorski.

I would like to define the notion of a prefoliation of a $\mathbb{K}$-differential space and to present theorems which describe some properties of leaves of the prefoliation.

First, I remind the definition of a $\mathbb{K}$-differential space.

Let $\mathbb{K}$ be an arbitrary field with a non-trivial norm (we can think here about $\mathbb{R}$ or $\mathbb{C}$). Let $M^{(0)} = \{\alpha : \quad \alpha : D_\alpha \to \mathbb{K}\}$ be a family of functions with an arbitrary family of sets $\{D_\alpha\}$ as domains. Define the set

$$.M^{(0)} = \bigcup_{\alpha \in M_0} D_\alpha$$

which will be called *the set of points of* $M^{(0)}$. In the set of points of $M^{(0)}$ define a topology $topM^{(0)}$ as the weakest topology containing the family

$$\{\alpha^{-1}(B) : B \text{ is open in } \mathbb{K} \text{ and } \alpha \in M^{(0)}\}.$$

Next set

$$anM^{(0)} := \begin{array}{l} \{\varphi \circ (\alpha_1, ...\alpha_m) : m \in \mathbb{N} \text{ and } \alpha_1, ..., \alpha_m \in M^{(0)} \\ \text{and } \varphi \text{ is an analytical function defined on an open} \\ \text{set in } \mathbb{K}^m \text{ with values in } \mathbb{K}\}. \end{array}$$

If $A \subset .M^{(0)}$ then

$$M^{(0)}|A := \{\alpha|A \cap D_\alpha : \alpha \in M^{(0)}\}$$

and

$$M_A^{(0)} := \{\beta : \forall_{p \in D_\beta} \exists_{U \in topM^{(0)}} \exists_{\alpha \in M^{(0)}} (p \in U \cap A \subset D_\beta \wedge U \subset D_\alpha \wedge \beta|U \cap A = \alpha|U \cap A)\}.$$

It is easy to see that $M^{(0)}|A \subset M_A^{(0)}$.

Definition 1.1. *The family M of functions with its values in $\mathbb{K}$ is called a $\mathbb{K}$-differential space, if the condition*

$$anM = M = M_{.M}$$

is fulfilled.

Let $M^{(0)}$ be an arbitrary family of functions with its values in $\mathbb{K}$. One can prove the following

Proposition 1.2. *The family $M := (anM^{(0)})_{.M^{(0)}}$ is the smallest $\mathbb{K}$-differential space with the set $.M^{(0)}$ as the set of points, containing $M^{(0)}$.*

Definition 1.3. *The $\mathbb{K}$-differential space M defined above is called the $\mathbb{K}$-differential space generated by a family $M^{(0)}$.*

Prof. W. Waliszewski has proved the following

Proposition 1.4. *Let M be a $\mathbb{K}$-differential space and $A \subset .M$. Then $M_A \subset M$ if and only if $A \in topM$.*

The above proposition gives

Corollary 1.5. *Let M be a $\mathbb{K}$-differential space and $A \in topM$. Then $M|A = M_A$.*

Let M, N be $\mathbb{K}$-differential spaces.

Definition 1.6. *The mapping $f : .M \to .N$ is said to be smooth if for each $\beta \in N$ we have $\beta \circ f \in M$.*

If the above condition holds then we write $f : M \to N$.

It is obvious that if $f : M \to N$ then $f : topM \to topN$ i.e. f is a continuous mapping respective to the topologies $topM$ and $topN$.

Let M be a $\mathbb{K}$-differential space and $p \in .M$. Define $M(p) := \{\alpha \in M : p \in D_\alpha\}$.

Definition 1.7. *Any $\mathbb{K}$-linear mapping $v : M(p) \to \mathbb{K}$ such that*

$$v(\alpha \cdot \beta) = v(\alpha) \cdot \beta(p) + \alpha(p) \cdot v(\beta)$$

for $\alpha, \beta \in M(p)$ is called a vector tangent to M at p. The family of all vectors tangent to M at p forms a vector space. This vector space is said to be tangent to M at p. It is denoted by T_pM.

It is easy to see that if $v \in T_pM$, $U \in topM$ and $\alpha \in M$ then $v(\alpha) = v(\alpha|U)$.

Let M, N be $\mathbb{K}$-differential spaces and $f : M \to N$. For each $p \in .M$, the mapping f determines a linear mapping $(f_*)_p : T_pM \to T_{f(p)}N$ called *a tangent mapping.* Namely, for $v \in T_pM$ and $\beta \in N(f(p))$ we have

$$((f_*)_p v)(\beta) = v(\beta \circ f).$$

Definition 1.8. *The mapping $f : M \to N$ is called an immersion, if for each $p \in .M$ the tangent mapping $(f_*)_p$ is a monomorphism.*

Now we define a prefoliation of a $\mathbb{K}$-differential space. Let $M = \{\alpha : \quad \alpha : D_\alpha \to \mathbb{K}\}$ be a $\mathbb{K}$-differential space.

Definition 1.9. *A pair (M, F) of $\mathbb{K}$-differential spaces is called a prefoliation of M if*

1) $.F = .M$,
2) $topF$ *is locally connected,*
3) $\forall_{p \in .M} \exists_{U \in topF} (p \in U \wedge F_U = M_U)$.

Connected components of $topF$ are called leaves of (M, F).

It is easy to see that the notion of a prefoliation is a generalization of the notion of the regular foliation, and of the Stefan foliation. Moreover, if (M, F) is a prefoliation of $\mathbb{K}$-differential space M then $(topM, topF)$ is a topological foliation in the sense of Ehresmann.

It is not difficult to prove the following

Theorem 1.10. *If (M, F) is a prefoliation of a $\mathbb{K}$-differential space M then for the mapping $f = id_{.M}$ we have $f : F \to M$.*

Proof. If $\beta \in M$ then for each $p \in .M$ denote by U_p such a neighbourhood of p respective to *topF* for which

$$F_{U_p} = M_{U_p}.$$

Thus we get an open covering $\{U_p\}_{p \in .M}$ of the set $.M$ respective to *topF* with

$$\beta | U_p \in M | U_p \subset M_{U_p} = F_{U_p}.$$

Therefore $\beta \in F$, since F is a $\mathbb{K}$-differential space. ■

¿From the theorem we get

Corollary 1.11. *Let (M, F) be a prefoliation of a $\mathbb{K}$-differential space M. Then $topM \subset topF$ and $M \subset F$.*

We also have

Theorem 1.12. *If (M, F) is a prefoliation of a $\mathbb{K}$-differential space M then the mapping $f = id_{.M}$ is an immersion.*

Proof. Suppose that $(f_*)_p(v) = 0$ for some $v \in T_pF$. Then for each $\beta \in M(p)$ we have $v(\beta \circ f) = 0$, i.e. for each $G \in topF$ with $p \in G$ we have

$$v((\beta \circ f)|G) = 0 \tag{1.1.1}$$

Let $\alpha \in F(p)$. There exists $U \in topF$ such that $p \in U$ and $F_U = M_U$ by the definition of a prefoliation. Therefore, $\alpha | D_\alpha \cap U \in F|U = F_U = M_U$ by Corollary 5. Thus there exist $V \in topM$ and $\gamma \in M$ such that $p \in V \cap U \subset D_\alpha$ and $V \subset D_\gamma$ and $\gamma | V \cap U = \alpha | V \cap U$. Obviously $\gamma \in M(p)$ and because of Corollary 11, $V \cap U \in topF$. Consequently,

$$v(\alpha) = v(\alpha | V \cap U) = v(\gamma | V \cap U) = v(\gamma \circ f | V \cap U) = 0$$

by (1.1.1). Thus $v = 0$. ■

Corollary 1.13. *If L is a leaf of a prefoliation (M, F) and $\varphi : L \ni q \mapsto q \in .M$ then $\varphi : F_L \to M$ and φ is an immersion.*

Corollary 1.14. *If L is a leaf of a prefoliation (M, F) then $M_L \subset F|L$.*

We show that even if $id : F \to M$ is an immersion and $(topM, topF)$ is a topological foliation then (M, F) has not to be a prefoliation.

Example 1.15. *Let M be the $\mathbb{R}$-differential space generated by the family of all continuous functions defined on $\mathbb{R}$ with values in $\mathbb{R}$ and let F be the $\mathbb{R}$-differential space generated by the family of all C^∞ functions defined on $\mathbb{R}$ with values in $\mathbb{R}$. Then $(topM, topF)$ is the trivial topological foliation in the sense of Ehresmann of $\mathbb{R}$. Remark that $f = id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is a smooth mapping $F \to M$ since $M \subset F$. One can prove that T_pF is a vector space of dimension 0 for each $p \in \mathbb{R}$. Therefore, $(f_*)_p$ is a monomorphism for each $p \in \mathbb{R}$.*

It is obvious that for each $U \in topF = topM$ we have $M_U = M|U \neq F|U = F_U$ which means that (M, F) is not a prefoliation of an $\mathbb{R}$-differential space M.

Using the definition of a prefoliation, one can prove

Lemma 1.16. *If (M, F) is a prefoliation of a $\mathbb{K}$-differential space, then for each $\beta \in F$ and for each $p \in D_\beta$ there exist $U \in topF$ and $\alpha \in M$ such that $p \in U \subset D_\beta$ and $\beta|U = \alpha|U$.*

Definition 1.17. *The leaf L of a prefoliation (M, F) is said to be proper if $(topF)|L = (topM)|L$.*

We have the following

Theorem 1.18. *Let (M, F) be a prefoliation of a $\mathbb{K}$-differential space M and L be a proper leaf of (M, F). Then*

$$M_L = F|L.$$

Proof. The inclusion $M_L \subset F|L$ holds for each L by Corollary 14.

Let $\gamma \in F|L$. By a local connectedness of $topF$ we have $L \in topF$ and consequently $\gamma \in F|L \subset F$ by Proposition 4 and Corollary 5. By Lemma 16, for each $p \in D_\gamma$ there exists $V \in topF$ such that $p \in V \subset D_\gamma \subset L$ and there exists $\alpha \in M$ such that $\gamma|V = \alpha|V$. Since L is proper, there exists $W \in topM$ such that $V = W \cap L$. Let $p \in D_\gamma$ and define $U := W \cap D_\alpha \in topM$ and $\overline{\alpha} := \alpha|U \in M$. Then

1) $p \in U \cap L \subset D_\gamma$ since $U \cap L \subset W \cap L = V \subset D_\gamma$;

2) $U \subset D_{\overline{\alpha}}$ (in fact, the equality holds);

3) $\overline{\alpha}|U \cap L = \alpha|U \cap L = \alpha|W \cap L \cap D_\alpha = \gamma|W \cap L \cap D_\alpha = \gamma|U \cap L$ since $W \cap L \cap D_\alpha \subset V$.

By 1)-3) we have $\gamma \in M_L$. ■

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SUBMODULES OF VECTOR FIELDS AND ALGEBROIDS

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Abstract

Algebroids and generalized algebroids defined in [3], particularly Courant algebroids considered in [1], define involutive and finite generated submodules of vector fields. But the most known algebroids are the Lie algebroids. They are considered by J. Pradines in [4] in connection with Lie groupoids, giving a coherent generalization of Lie theory. The third theorem of Lie related to the integration of Lie algebroids to Lie groupoids failed from the global viewpoint (Almeida-Molino, 1985, see [2, Theorem 4.4]). Most of people focused in particular to the global integration problem of Lie algebroids to Lie groupoids. Even the integration is not always possible, some particular integrable cases or obstructions regarding the integration were studied.

The algebroids considered in this paper are vector bundles which the module of sections fulfills the conditions of a Lie algebroid, except the Jacobi condition. But as our knowledge is, the relation between algebroids and vector fields, regarding the situation when a submodule of vector fields can be defined by the image of an algebroid, has not been yet studied. We prove that *the necessary and sufficient condition for a submodule of vector fields to be defined by the image of an algebroid is that the module be involutive and finite generated*. It means that the algebroid is a sufficient notion for these modules. As a first case when this situation occurs, we prove that *a singular Riemannian foliation is always defined by an algebroid*. As a second application, the canonical central anchored bundle of regular and singular Riemannian foliations is defined and the cojecture of P. Molino which asserts that *the closures of leaves of a singular Riemannian foliation is also a singular Riemannian foliation* is proved. In fact, Molino has left to prove only that the closures of leaves of a singular Riemannian foliation is a Stefan-Sussmann foliation [2, Chapter 6]; here we fill up this gap.

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MODULAR CLASSES OF ANCHORED MODULES

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The categories of modules with differentials and the vector bundles with differentials, are defined by the first author in [5]. As explained in [4] or [5], they are two categories of vector bundles and two categories of modules; they are also two functors from the each category to the other. Some corresponding functors are induced from the categories of vector bundles with differentials to the corresponding categories of modules with differentials.

As it is proved in [7], there are two functors (called here the *derived functors*) from the two categories of anchored modules which allow linear connections to the two categories of Lie pseudoalgebras; in the presence of linear connections, these functors can be defined on all the categories of modules with differentials. A Lie pseudoalgebra (the derived Lie pseudoalgebra) can be associated with an anchored module which allows a linear connection. The derived Lie pseudoalgebra does not depend on the linear connection, but on the anchor, being an invariant object associated with an anchored module which allows a linear connection. Considering also the correspondences of morphisms, one define two pseudofunctors (i.e. a functor except sending the identity in the identity) and two natural functors (called the derived functors) respectively on the two categories of anchored modules which allow linear connections to the corresponding two categories of Lie pseudoalgebras.

We prove in the paper that the construction of the derived functors can be also performed using an other way, which is more suitable for vector bundles [6].

We show that a linear connection lifts on the derived module to a curvature free connection. The Picard groups related to anchored modules as well as the modular classes of preinfinitesimal modules and almost Lie structures are defined. The modular classes defined here agrees in a certain sense with the modular class of a Lie pseudoalgebra defined by J. Huebschmann [3].

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ERGODIC AND SPECTRAL PROPERITIES OF LAGRANGIAN AND HAMILTONIAN DYNAMICAL SYSTEMS AND THEIR ADIABATIC PERTURBATIONS

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Abstract

Any Lagrangian function on a closed finite-dimensional manifold M, when depending 2π–periodically on the evolution parameter generates so called Lagrangian flow. Its related group of diffeomorphisms on $T(M) \times \mathbb{S}^1$ makes it possible to construct the set of normed (probabilistic) invariant measures on $T(M) \times \mathbb{S}^1$. The latter appears to be a convex set completely characterized by means of so called extreme points being at the same time due to a result of J. Mather ergodic measures of the Lagrangian flow under regard. On the other hand, there exists a natural mapping from the space of all invariant measures space mentioned above into the first homology group $H_1(M;\mathbb{R})$ of the manifold M via a well known Mather's construction and some its generalizations subject to nonautonomous Hamiltonian flows on symplectic speces, whose image is exactly the measure homology of our Lagrangian or the corresponding Hamiltonian system. Its properties prove to be very important for detecting the corresponding ergodic measures, making use a new tool of its studying related with so called Legendrian transformations and Poincare -Cartan invariants. Moreover in the case when our Lagrangian function depends adiabatically on a small parameter $\varepsilon\downarrow 0$ through the expression $\varepsilon t \in \mathbb{R}/2\pi\mathbb{Z}$, a suitable application of the Legendrian transformation together with the technique of Poincare -Cartan invariants makes it possible to investigate the existence and properties of so called adiabatic invariants and the corresponding limiting ergodic measures on $T(M) \times \mathbb{S}^1$. These same properties can be studied simultaneously making use also of the theory of spectral invariants applied to the generator of the corresponding Hamiltonian flow on the symplectic phase space $T^*(M)$.

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INFINITE DIMENSIONAL LIE THEORY BY MEANS OF THE EVOLUTION MAPPING

TOMASZ RYBICKI

Abstract

An infinite dimensional Lie theory is said to be abstract if in the definition of the smooth structure on Lie groups charts are not required. Several abstract settings exist (Souriau, Chen, Omori), but usually they do not correspond to each other. The necessity of them is motivated by important examples and applications.

An infinite dimensional Lie group G with its Lie algebra $\mathfrak{g}$ is called regular if there is a bijective evolution mapping

$$\mathrm{evol}_G^r : C^\infty(\mathbb{R}, \mathfrak{g}) \to \mathfrak{C}^\infty((\mathbb{R}, \mathfrak{o}), (\mathfrak{G}, \mathfrak{e}))$$

such that its evaluation at $1 \in \mathbb{R}$ is smooth. This notion has been introduced by Milnor. A concept generalizing regular Lie groups is proposed. In this concept the smooth structure is defined by means of the evolution mapping evol_G^r. Basic properties of Lie groups can be derived from our definition. New interpretations of the inheritance property, the third Lie theorem, and other facts are possible.

KEYWORDS: infinite dimensional Lie group, regularity

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ON THE SET OF GEODESIC VECTORS OF A LEFT-INVARIANT METRIC

JÁNOS SZENTHE

If $(M, <, >)$ is a Riemannian manifold its geodesic $\gamma : R \to M$ is said to be homogeneous if there is a 1-parameter group of isometries $\Phi : R \times M \to M$ of the Riemannian manifold such that

$$\gamma(\tau) = \Phi(\tau, \gamma(0)), \ \tau \in \mathbb{R}$$

holds. As the comprehensive paper of C. S. Gordon shows the existence of homogeneous geodesics in homogeneous Riemannian manifolds has essential geometric consequences [G]. Several results have been obtained recently concerning the existence of homogeneous geodesics. First it has been shown by V. V. Kajzer that if G is a Lie group and $<, >$ a left-invariant Riemannian metric on G then the Riemannian manifold $(G, <, >)$ has at least 1 homogeneous geodesic [Ka]. Generalizing this result of Kajzer it has been shown by O. Kowalski and J. Szenthe that if $M = G/H$ is a homogeneous manifold and $<, >$ an invariant metric on G/H then the homogeneous Riemannian manifold $(G/H, <, >)$ has at least 1 homogeneous geodesic [Ko-Sz]. Moreover, it has been shown by Szenthe that if G is a compact semi-simple Lie group of $rank \geq 2$ and $<, >$ is a left-invariant Riemannian metric on G then the Riemannian manifold has infinitely many homogeneous geodesics [Sz]. In the study of the set of homogenous geodesics of a homogeneous Riemannian manifold $(G/H, <, >)$ the concept of geodesic vector proved to be convenient [Ko-V]. Let $\Phi : G \times (G/H) \to G/H$ be the canonical action, g the Lie algebra of G and $Exp : g \to G$ its exponential map. Put $o = H \in G/H$, fix a tangent vector $v \in T_o(G/H)$ and consider the geodesic $\gamma : R \to G/H$ defined by $v = \dot{\gamma}(0)$. It is said that v is a geodesic vector if γ is a homogeneous geodesic of $(G/H, <, >)$; in other words if

$$\gamma(\tau) = \Phi(Exp(\tau X), o), \ \tau \in \mathbb{R}$$

holds with some $X \in g$. The study of the set of homogeneous geodesics of a homogenous Riemannian manifold is obviously reducible to the study of the set of its homogeneous vectors. However, it seems that the set of the geodesic vectors of a homogeneous Riemannian manifold does not admit a simple description in general. Namely, O. Kowalski, S. Nikčević and Z. Vlašek in a joint paper have given several examples where the set of geodesic vectors have essentially different structure. In the lecture results are presented concerning the set of geodesic vectors of a homogeneous Riemannian manifold $(G, <, >)$, where G is a compact semi-simple Lie group and $<, >$ is a left-invariant Riemannian metric on G.

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ON THE STABILITY OF SMOOTH DYNAMICAL SYSTEMS AND DIFFEOMORPHISMS

ANDRZEJ ZAJTZ

Abstract

General methods of studying global stability problems of smooth dynamical systems and diffeomorphisms are presented.

In particular, one proves that any complete C^∞ vector field X in a Hilbert space E satisfying $\langle X(x), v\rangle \geq \delta > 0$ for some constant field v and all $x \in E$ (for instance, if $\|X - v\| \leq \frac{\|v\|}{2}$) is globally rectifiable to v by a C^∞ diffeomorphism of E. Thus any nonzero constant vector field in E is smoothly stable in a 0-order neighborhood.

Similarly one obtains the global stability (in a 1st-order neighborhood) of expansive non-resonant linear systems.

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